

QCD₃ and the Replica Method

G. AKEMANN ^a, D. DALMAZI ^{b*}, P.H. DAMGAARD ^c and J.J.M. VERBAARSHOT ^b

^a Max-Planck-Institut für Kernphysik
Saupfercheckweg 1
D-69117 Heidelberg
Germany

^b Dept. of Physics and Astronomy, SUNY
Stony Brook, N.Y. 11794
U.S.A.

^c The Niels Bohr Institute
Blegdamsvej 17
DK-2100 Copenhagen Ø
Denmark

Abstract

Using the replica method, we analyze the mass dependence of the QCD₃ partition function in a parameter range where the leading contribution is from the zero momentum Goldstone fields. Three complementary approaches are considered in this article. First, we derive exact relations between the QCD₃ partition function and the QCD₄ partition function continued to half-integer topological charge. The replica limit of these formulas results in exact relations between the corresponding microscopic spectral densities of QCD₃ and QCD₄. Replica calculations, which are exact for QCD₄ at half-integer topological charge, thus result in exact expressions for the microscopic spectral density of the QCD₃ Dirac operator. Second, we derive Virasoro constraints for the QCD₃ partition function. They uniquely determine the small-mass expansion of the partition function and the corresponding sum rules for inverse Dirac eigenvalues. Due to de Wit-'t Hooft poles, the replica limit only reproduces the small mass expansion of the resolvent up to a finite number of terms. Third, the large mass expansion of the resolvent is obtained from the replica limit of a loop expansion of the QCD₃ partition function. Because of Duistermaat-Heckman localization exact results are obtained for the microscopic spectral density in this way.

NBI-HE-00-38
SUNY-NTG-00/54
hep-th/0011072

*On leave from UNESP (Guaratinguetá-Brazil)

1 Introduction

QCD in 2+1 space-time dimensions (QCD₃) provides an interesting alternative to spontaneous chiral symmetry breaking in 3+1 dimensions. Because of the absence of a γ_5 matrix in any odd number of space-time dimensions, ordinary chiral symmetry cannot be defined. Yet, for an even number of flavors $2N_f$ it has been known for some time that there exists a close analogue of chiral symmetry in 2+1 dimensions (see, *e.g.*, ref. [1]). In this case the global flavor symmetry may break spontaneously according to $U(2N_f) \rightarrow U(N_f) \times U(N_f)$ [1, 2]. For an odd number of flavors it has been argued that the pattern of spontaneous symmetry breaking is replaced by $U(2N_f+1) \rightarrow U(N_f+1) \times U(N_f)$ [2]. This picture of spontaneous flavor symmetry breaking has been supported by recent Monte Carlo simulations [3].

Spontaneous flavor symmetry breaking such as discussed above gives rise to one Goldstone boson for each broken generator. In ref. [2] the low-energy effective partition function of the corresponding chiral Lagrangian was used to extract information on the spectrum of the Dirac operator of QCD₃. It was also suggested [2] that Random Matrix Theory may be used to exactly compute the microscopic properties of the Dirac spectrum in much the same way as was first proposed for QCD in 3+1 dimensions (QCD₄) [4]. The resulting spectral correlation functions for the ensemble relevant for QCD₃ (a variant of the Unitary Ensemble, UE) are indeed universal in Random Matrix Theory [5, 6, 7]. They can be expressed entirely in terms of (2+1)-dimensional finite-volume partition functions [8]. Recently, the patterns of flavor symmetry breaking for theories with gauge group SU(2) or with fermions in the adjoint representation of SU(N_c) in 2+1 dimensions have been shown to correspond to two other classes of the classical Random Matrix Theories [9], namely the Orthogonal Ensemble (OE) and the Symplectic Ensemble (SE), respectively, in precise analogy to the (3+1)-dimensional case [10]. The microscopic spectral densities of these theories were calculated recently in [11, 12].

To derive non-perturbative analytical results for the smallest eigenvalues of the Dirac operator it is important to *prove* that universal Random Matrix Theory results coincide with the low-energy limit of the field theory. For QCD₄ this has been achieved for the spectral density and the two-point spectral correlation function by means of the so-called supersymmetric method [13, 14, 15, 16, 17, 18, 19, 20]. In this method, one adds a set of fermionic and bosonic ghost quarks (or “valence quarks”, in a terminology borrowed from lattice gauge theory [21, 22, 23]) such that their determinants cancel for equal masses. In this way one obtains a generating functions for the resolvent (the partially quenched chiral condensate in the language of lattice gauge theory) or the higher order correlations functions. The spectral density then follows from the discontinuity of the resolvent in the complex mass plane of the ghost quarks [13].

An alternative method to eliminate the determinants of ghost quarks is the replica method [24]. It is based on adding either n fermionic or n bosonic valence quarks (also called replicas), performing an analytical continuation in n and taking the limit $n \rightarrow 0$ at the end of the calculation. Although the replica method has been challenged [25], this method has received a considerable boost in the context of condensed matter physics during the past year [26, 27, 28, 29]. Recently, the replica method has also been applied to the field theory corresponding to the low-energy limit of QCD₄ [30, 31, 32]. The advantage of the fermionic replica method is that we are familiar with the pattern of spontaneous chiral symmetry breaking, and it is straightforward to define the corresponding low-energy field theory. Although we have less experience with spontaneous symmetry breaking in the supersymmetric formulation, the consensus is that such internal supersymmetries are not spontaneously broken. Based on this assumption there should not be any problems in defining the low-energy limit of the supersymmetric partition function of this theory, which in turn is the generating function of the resolvent.

This has recently been confirmed by the explicit calculation of Szabo [33], who has demonstrated how the microscopic spectral density and other spectral correlation functions of QCD₃ can be obtained in this way. Nevertheless, precisely in the case of QCD₃ it is advantageous to explore also the replica method. The reason is as follows. In general, one expects that the replica method can only provide us with asymptotic series for spectral correlation functions of the Dirac operator, and not with exact results. In QCD₃, however, the large mass asymptotic expansion of the resolvent terminates (it is semiclassically exact [29] in the quenched case) and exact results will be obtained this way.

An important ingredient of this paper is the discussion of general relations between the finite-volume partition functions for QCD in 2+1 and 3+1 dimensions. They will be introduced in section 2 and the relations will be formulated and proved in section 3. These relations hold in all generality and are independent of any detailed derivations in any of the two theories. The implications of these relations are discussed in section 4, where we use the replica method to derive our main result: two completely general relations between the microscopic spectral densities of the two theories. Thus, once we know the spectral densities of QCD₄, the spectral densities of QCD₃ follow. In section 4.1 it is shown that results for the microscopic spectral density and microscopic spectral correlation functions obtained in this way are in exact agreement with previous results from Random Matrix Theory [2]. An exact calculation of the resolvent of the QCD₃ partition function from its finite volume partition function is thus possible by using the recent results for the QCD₄ finite volume partition function [31] (see section 4.2). In sections 5 and 6 we perform explicit calculations in QCD₃ based on the replica method. First, in section 5, a small-mass series expansion of the resolvent of the QCD₃ Dirac operator is generated by means of Virasoro constraints on the finite volume QCD₃ partition function. As a by-product of this analysis we derive a series of spectral sum rules for the Dirac operator in QCD₃. In section 6, we explore the (large-mass) saddle-point expansion of the resolvent, which turns out to *truncate* after a finite number of terms, and hence is exact. This truncation is consistent with the general relations to the QCD₄ partition function derived in section 3. Our results are summarized in section 7.

2 The QCD₃ Partition Function

The Dirac operator of the QCD₃ partition function is given by

$$D = i(\partial_k + iA_k)\sigma_k, \quad (2.1)$$

where the σ_k are the Pauli matrices and the A_k are the $SU(N_c)$ gauge fields. The partition function for $2N_f$ flavors is defined by

$$\mathcal{Z} = \langle \det(D\mathbf{1}_{2N_f} + iM) \rangle, \quad (2.2)$$

where M is a Hermitian mass matrix and the average is over the gauge field configurations weighted by the Yang-Mills action. Without loss of generality, the mass matrix can always be chosen diagonal, and we will only consider this case. Generally, the fermion determinant in QCD₃ is not positive definite, resulting in a Chern-Simons term due to the anomaly [34]. However, for quark masses occurring in pairs m_k and $-m_k$, the fermion determinant is positive definite, and the theory is anomaly free. In that case the sign of the $\langle \bar{\psi}_k \psi_k \rangle$ is determined by the sign of the quark mass,

$$\langle \bar{\psi}_k \psi_k \rangle = -\frac{m_k}{|m_k|} \Sigma. \quad (2.3)$$

The low-energy effective partition function is constructed from the requirements that its transformation properties should reflect those of QCD₃, i.e. the partition function is invariant under $U \in U(2N_f)$ transformations of the quark fields if, at the same time, the mass matrix is transformed according to

$$M \rightarrow U M U^{-1}. \quad (2.4)$$

The chiral condensate given by (2.3) is only invariant under an $U(N_f) \times U(N_f)$ subgroup of $U(2N_f)$. The Goldstone manifold is thus given by $U(2N_f)/U(N_f) \times U(N_f)$. The matrices in this manifold transform according to

$$U \rightarrow U_1 U U_1^{-1}, \quad U_1 \in U(2N_f). \quad (2.5)$$

To lowest order in the mass, the mass term of the effective theory is given by

$$\mathcal{L}_m = \pm \text{Re} \Sigma \text{Tr} M U. \quad (2.6)$$

Because of the integration over the coset in the partition function, the overall sign of the mass term is irrelevant. For an even number of flavors in the limit where the Compton wavelength of the pseudo-Goldstone bosons is much larger than the size of the box, the partition function factorizes into a zero momentum part and a non-zero momentum part. The first one, which is also known as the QCD₃ finite volume partition function, is thus given by [2]

$$\mathcal{Z}_{\text{QCD}_3}^{(2N_f)}(\{\mu_i\}) = \int_{U(2N_f)} dU \exp[V \Sigma \text{Tr}(\mathcal{M} U \Gamma_5 U^\dagger)], \quad (2.7)$$

with $\mathcal{M} = \text{diag}(m_1, \dots, m_{N_f}, -m_1, \dots, -m_{N_f})^1$, $\Gamma_5 = \text{diag}(\mathbf{1}_{N_f}, -\mathbf{1}_{N_f})$ and V the volume of the three-dimensional space-time. We have used an explicit representation of the coset, $U \rightarrow U \Gamma_5 U^\dagger$, and extended the integration to the full $U(2N_f)$ group. This gives only rise to an overall volume factor. The partition function (2.7) is only a function of the rescaled masses

$$\mu_i \equiv m_i V \Sigma, \quad (2.8)$$

which will be kept fixed as $V \rightarrow \infty$. We shall here consider QCD₃ at fixed ultraviolet cut-off Λ . In the infrared the theory is regularized by the three-volume V , which is only eventually taken to infinity. If not only the volume is finite but also space-time is discretized such as in lattice QCD, the total number of Dirac eigenvalues is finite and will be denoted by \mathcal{N} .

For an odd number of flavors one of the quarks is unpaired, and the fermion determinant may change sign under large gauge transformations. In this case the sign of the chiral condensate is not given by the sign of the quark mass. There are two inequivalent possibilities for the vacuum state, determined by the sign of the fermion determinant. They are not connected by unitary transformations of the coset elements. In a regime for which $m \sim 1/V\Sigma$, both states have to be included in the low energy limit of the QCD partition function. The QCD₃ partition function satisfies the parity relation

$$\mathcal{Z}_{\text{QCD}_3}(-m) = (-1)^{\mathcal{N}N_f} \mathcal{Z}_{\text{QCD}_3}(m). \quad (2.9)$$

For an odd number of flavors the low-energy effective partition function is thus given by

$$\mathcal{Z}_{\text{QCD}_3}^{(2N_f+1)}(\{\mu_i\}) = \int_{U(2N_f+1)} dU \cosh[V \Sigma \text{Tr}(\mathcal{M} U \tilde{\Gamma}_5 U^\dagger)], \quad (2.10)$$

¹ Recall that in 2+1 dimensions there is no definite sign for the fermionic mass term, a consequence of the fact that in any odd number of space-time dimensions the two sets of γ -matrices $\{\gamma_i\}$ and $\{-\gamma_i\}$ are inequivalent irreducible representations of the Clifford algebra.

for even number of eigenvalues \mathcal{N} , and

$$\mathcal{Z}_{\text{QCD}_3}^{(2N_f+1)}(\{\mu_i\}) = \int_{U(2N_f+1)} dU \sinh[V\Sigma\text{Tr}(\mathcal{M}U\tilde{\Gamma}_5U^\dagger)] , \quad (2.11)$$

for odd number of eigenvalues. The second partition function (2.11) is not positive definite, and is unsuitable as a physical partition function. In particular, it vanishes when the unpaired fermion mass vanishes [7]. Nevertheless, this second partition function can appear at intermediate stages [8].

These partition functions are to be compared with the QCD₄ finite volume partition function given by [4]

$$\mathcal{Z}_\nu^{(N_f)}(\{\mu_i\}) = \int_{U(N_f)} dU (\det U)^\nu \exp \left[\frac{1}{2} V \Sigma \text{Tr}(\mathcal{M}U^\dagger + U\mathcal{M}^\dagger) \right] , \quad (2.12)$$

where ν is the topological charge, and $\mathcal{M} = \text{diag}(m_1, \dots, m_{N_f})$. In both cases Σ stands for the infinite-volume quark-antiquark condensate. As we will see in the next section, there are deep relations between these two types of group integrals. Although the structural nature of these relations suggest that they can be obtained from general group theoretical arguments, we have only been able to prove them at the technical level.

3 The QCD₃–QCD₄ Connection

In this section we shall prove a set of surprising relations between group integrals of Itzykson-Zuber kind (2.7), (2.10) and (2.11), and group integrals of “external field” type (2.12). Some of these relations were already pointed out in ref. [35], based on the relationship to Random Matrix Theory, but we shall here prove them directly from the finite volume partition functions.

Both of the group integrals in eqs. (2.7) and (2.12) can be expressed in closed form. The first is simply an Itzykson-Zuber integral [36] with a particular N_f -degeneracy in one of the matrices (Γ_5), and the result is [6]²

$$\mathcal{Z}_{\text{QCD}_3}^{(2N_f)}(\{\mu_i\}) = (-1)^{N_f(N_f+1)/2} \frac{\det \begin{pmatrix} \mathbf{A}(\{\mu_i\}) & \mathbf{A}(\{-\mu_i\}) \\ \mathbf{A}(\{-\mu_i\}) & \mathbf{A}(\{\mu_i\}) \end{pmatrix}}{\Delta(\mathcal{M})} , \quad (3.1)$$

where the $N_f \times N_f$ matrix $\mathbf{A}(\{\mu_i\})$ is defined by

$$\mathbf{A}(\{\mu_i\})_{jl} \equiv (\mu_j)^{l-1} e^{\mu_j} , \quad j, l = 1, \dots, N_f . \quad (3.2)$$

and the Vandermonde determinant is taken over all masses (including the sign-mirrored ones),

$$\Delta(\mathcal{M}) = \prod_{i>j}^{2N_f} (\mu_i - \mu_j) . \quad (3.3)$$

Recently, it was shown that the above partition function can be expressed much more compactly by means of just a single $N_f \times N_f$ determinant [33]. However, precisely for our present purposes the above form is actually more convenient.

²Note that we have chosen a different prefactor than in ref. [6] in order to make the partition function positive definite. For averages the difference is of course irrelevant.

For an odd number of flavors the integral (2.10) and (2.11) can also be evaluated as Itzykson-Zuber integrals. The result is

$$\mathcal{Z}_{\text{QCD}_3^\pm}^{(2N_f+1)}(\mu, \{\mu_i\}) = (-1)^{N_f(N_f+3)/2} \frac{2^{N_f}}{\Delta(\mathcal{M})} \frac{1}{2} \left[\det \mathbf{D}(\mu, \{\mu_i\}) \pm (-1)^{N_f} \det \mathbf{D}(-\mu, \{-\mu_i\}) \right], \quad (3.4)$$

where the $(2N_f + 1) \times (2N_f + 1)$ matrix \mathbf{D} is defined as

$$2^{N_f} \det \mathbf{D}(\mu, \{\mu_i\}) \equiv \det \begin{pmatrix} \mathbf{A}(\mu, \{\mu_i\})_{N_f+1 \times N_f+1} & \mathbf{A}(-\mu, \{-\mu_i\})_{N_f+1 \times N_f} \\ \mathbf{A}(\{-\mu_i\})_{N_f \times N_f+1} & \mathbf{A}(\{\mu_i\})_{N_f \times N_f} \end{pmatrix}. \quad (3.5)$$

Here, the matrix \mathbf{A} is given by eq. (3.2), with the corresponding quadratic or rectangular size explicitly indicated.

For integer ν the integral (2.12) can be expressed as [37],

$$\mathcal{Z}_\nu^{(N_f)}(\{\mu_i\}) = \det \mathbf{B}(\{\mu_i\}) / \Delta(\{\mu_i^2\}), \quad (3.6)$$

where the matrix \mathbf{B} in eq. (3.6) is given by

$$\mathbf{B}(\{\mu_i\})_{jl} = \mu_j^{l-1} I_\nu^{(l-1)}(\mu_j), \quad j, l = 1, \dots, N_f, \quad (3.7)$$

with $I_\nu^{(l-1)}(\mu) = (\partial/\partial\mu)^{l-1} I_\nu(\mu)$. The denominator is given by the Vandermonde determinant of squared masses,

$$\Delta(\{\mu_i^2\}) \equiv \prod_{i>j}^{N_f} (\mu_i^2 - \mu_j^2) = \det_{i,j} [(\mu_i^2)^{j-1}]. \quad (3.8)$$

In both cases the normalization convention of the integrals has been chosen for later convenience.

For non-integer values of ν we define the QCD_4 partition function according to the analytical continuation in ν of the Bessel functions [31],

$$I_\nu(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{z \cos \theta} e^{i\nu\theta} d\theta - \frac{\sin \nu\pi}{\pi} \int_0^\infty ds e^{-z \cosh s - \nu s}, \quad (3.9)$$

and not by analytical continuation in ν of the unitary matrix integral (2.12).

The integral (2.12) satisfies an important relation known as flavor-topology duality [38]. A proof that greatly simplifies earlier proofs [38] is obtained by realizing that the matrix \mathbf{B} in (3.6) can be replaced by

$$\mathbf{B}(\{\mu_i\})_{jl} \rightarrow \mu_j^{l-1} I_{\nu+l-1}(\mu_j), \quad j, l = 1, \dots, N_f, \quad (3.10)$$

If we take the limit $\mu \rightarrow 0$ and use the fact that $I_\nu(\mu) \sim (\mu/2)^\nu / \Gamma(\nu+1)$ we find the the flavor-topology relation

$$\lim_{\mu \rightarrow 0} \Gamma(\nu+1) \left(\frac{2}{\mu} \right)^\nu \mathcal{Z}_\nu^{(N_f+1)}(\mu, \{\mu_i\}) = \prod_{j=1}^{N_f} \frac{1}{\mu_j} \mathcal{Z}_{\nu+1}^{(N_f)}(\{\mu_i\}). \quad (3.11)$$

The general relation obtained by iterating (3.11) with respect to N_f is valid for arbitrary ν provided that $\mathcal{Z}_\nu^{(N_f)}$ is analytically continued in ν according to (3.9).

After these preliminary definitions, we are now ready to state and prove three theorems.

THEOREM I - Even number of flavors in QCD₃:

Let $\mathcal{Z}_{\text{QCD}_3}^{(2N_f)}(\{\mu_i\})$ and $\mathcal{Z}_\nu^{(N_f)}(\{\mu_i\})$ be as defined in eqs. (2.7) and (2.12), with the normalization conventions (3.1) – (3.8). Then the following identity holds:

$$\mathcal{Z}_{\text{QCD}_3}^{(2N_f)}(\{\mu_i\}) = \pi^{N_f} \mathcal{Z}_{\nu=-1/2}^{(N_f)}(\{\mu_i\}) \mathcal{Z}_{\nu=+1/2}^{(N_f)}(\{\mu_i\}) . \quad (3.12)$$

PROOF: We start by simplifying the right hand side, using that for $\nu = \pm 1/2$ we have

$$I_{\nu=-1/2}(\mu) = \sqrt{\frac{2}{\pi\mu}} \cosh(\mu) \quad \text{and} \quad I_{\nu=+1/2}(\mu) = \sqrt{\frac{2}{\pi\mu}} \sinh(\mu) . \quad (3.13)$$

One can easily convince oneself that in eq. (3.7) only terms with all derivatives acting on $\cosh(\mu_j)$ and $\sinh(\mu_j)$, respectively, contribute to the partition function. All other terms containing derivatives of the square roots can be eliminated by adding multiples of columns from the right, which leaves the determinant invariant. We can thus pull out the common factors of square roots and arrive at

$$\begin{aligned} \mathcal{Z}_{\nu=-1/2}^{(N_f)}(\{\mu_i\}) &= \prod_{j=1}^{N_f} \sqrt{\frac{2}{\pi\mu_j}} \det \mathbf{C}(\{\mu_i\}) / \Delta(\{\mu_i^2\}) , \quad \mathbf{C}(\{\mu_i\})_{jl} \equiv \mu_j^{l-1} \cosh^{(l-1)}(\mu_j) \\ \mathcal{Z}_{\nu=+1/2}^{(N_f)}(\{\mu_i\}) &= \prod_{j=1}^{N_f} \sqrt{\frac{2}{\pi\mu_j}} \det \mathbf{S}(\{\mu_i\}) / \Delta(\{\mu_i^2\}) , \quad \mathbf{S}(\{\mu_i\})_{jl} \equiv \mu_j^{l-1} \sinh^{(l-1)}(\mu_j) . \end{aligned} \quad (3.14)$$

Here and below $\cosh^{(p)}(\mu)$ or $\sinh^{(p)}(\mu)$ denotes the p -th derivative of $\cosh(mu)$ or $\sinh(mu)$. Turning next to the QCD₃ side, the Vandermonde determinant can be written as,

$$\Delta(\mathcal{M}) = (-1)^{N_f(N_f+1)/2} \prod_{k=1}^{N_f} 2\mu_k \Delta^2(\{\mu_i^2\}) . \quad (3.15)$$

In order to bring the $(2N_f) \times (2N_f)$ determinant in eq. (3.1) into the desired form we iteratively add and subtract columns to obtain cosh and sinh. Starting with the first column we add the N_f -th column to obtain $(2 \cosh(\mu_1), \dots, 2 \cosh(\mu_{N_f}), 2 \cosh(\mu_1), \dots, 2 \cosh(\mu_{N_f}))$. Then, subtracting $1/2$ of this new first column from the N_f -th column, we obtain $(-\sinh(\mu_1), \dots, -\sinh(\mu_{N_f}), \sinh(\mu_1), \dots, \sinh(\mu_{N_f}))$ as a new column vector at position N_f . The common factor of 2 can be taken out of the first column. We proceed in the same way with the second and $(N_f + 2)$ -nd column, where cosh and sinh get interchanged in the resulting columns compared to the previous step. After completing this alternating procedure we arrive at the following block structure for the QCD₃ side:

$$\begin{aligned} \mathcal{Z}_{\text{QCD}_3}^{(2N_f)}(\{\mu_i\}) &= (-1)^{N_f(N_f+1)/2} \frac{2^{N_f}}{\Delta(\mathcal{M})} \det \begin{pmatrix} \mathbf{C}(\{\mu_i\}) & -\mathbf{S}(\{\mu_i\}) \\ \mathbf{C}(\{\mu_i\}) & \mathbf{S}(\{\mu_i\}) \end{pmatrix} \\ &= (-1)^{N_f(N_f+1)/2} \frac{2^{N_f}}{\Delta(\mathcal{M})} \det \begin{pmatrix} 2\mathbf{C}(\{\mu_i\}) & \mathbf{0} \\ \mathbf{C}(\{\mu_i\}) & \mathbf{S}(\{\mu_i\}) \end{pmatrix} , \end{aligned} \quad (3.16)$$

where in the last step we have added the lower blocks to the upper ones, by a subsequent addition of rows. We can now perform a Laplace expansion into products of $N_f \times N_f$ blocks, which due to the zero block, $\mathbf{0}$, only yields one single product of two $N_f \times N_f$ determinants. Together with eq. (3.15) we obtain the final result:

$$\mathcal{Z}_{\text{QCD}_3}^{(2N_f)}(\{\mu_i\}) = \frac{2^{N_f}}{\Delta^2(\{\mu_i^2\}) \prod_{k=1}^{N_f} \mu_k} \det \mathbf{C}(\{\mu_i\}) \det \mathbf{S}(\{\mu_i\}) . \quad (3.17)$$

By comparing eqs. (3.17) and (3.14) we can read off the prefactor in the theorem eq. (3.12).

THEOREM II - Odd number of flavors in QCD₃:

Let in addition $\mathcal{Z}_{\text{QCD}_3^\pm}^{(2N_f+1)}(\mu, \{\mu_i\})$ be as defined in eqs. (3.4) and (3.5). Then the following two identities hold:

$$\mathcal{Z}_{\text{QCD}_3^\pm}^{(2N_f+1)}(\mu, \{\mu_i\}) = \pi^{N_f} \sqrt{\frac{\pi\mu}{2}} \mathcal{Z}_{\nu=\mp 1/2}^{(N_f+1)}(\mu, \{\mu_i\}) \mathcal{Z}_{\nu=\pm 1/2}^{(N_f)}(\{\mu_i\}) . \quad (3.18)$$

PROOF: Since the right hand side already follows from eq. (3.14) we start with the QCD₃ side, treating both cases, \pm , simultaneously. The Vandermonde determinant with one additional mass can be treated similarly to relation (3.15) and we obtain

$$\Delta(\mathcal{M}) = (-1)^{N_f(N_f+1)/2} \prod_{j=1}^{N_f} (\mu^2 - \mu_j^2) \prod_{l=1}^{N_f} 2\mu_l \Delta(\{\mu_i^2\})^2 . \quad (3.19)$$

Next, we proceed by adding and subtracting columns as in the proof of the previous theorem. We arrive at the same determinant with blocks of matrices \mathbf{C} and \mathbf{S} , except that the $(N_f + 1)$ -th columns remains unchanged. Here, we are dealing with an odd number of flavors and the determinant in eq. (3.4) is of size $(2N_f + 1) \times (2N_f + 1)$, with leaves one column unpaired. We thus have for eq. (3.5)

$$2^{N_f} \det \mathbf{D}(\mu, \{\mu_i\}) = 2^{N_f} \det \begin{pmatrix} \mu^{j-1} \cosh^{(j-1)}(\mu) & \mu^{N_f} e^\mu & -\mu^{j-1} \sinh^{(j-1)}(\mu) \\ \mathbf{C}(\{\mu_i\}) & \mu_i^{N_f} e^{\mu_i} & -\mathbf{S}(\{\mu_i\}) \\ \mathbf{C}(\{\mu_i\}) & (-\mu_i)^{N_f} e^{-\mu_i} & \mathbf{S}(\{\mu_i\}) \end{pmatrix} . \quad (3.20)$$

When adding the second determinant of eq. (3.4) with negative arguments we can use the fact that the matrix \mathbf{C} is even whereas \mathbf{S} is odd. After taking out all the minus signs from the matrix \mathbf{S} , which gives $(-1)^{N_f}$, we can add the two determinants $\det \mathbf{D}(\mu, \{\mu_i\})$ and $\det \mathbf{D}(-\mu, \{-\mu_i\})$, as they now differ only by the $(N_f + 1)$ -th column. The resulting determinant reads

$$\begin{aligned} \det \mathbf{D}(\mu, \{\mu_i\}) \pm (-1)^{N_f} \det \mathbf{D}(-\mu, \{-\mu_i\}) &= \\ &= \det \begin{pmatrix} \mu^{j-1} \cosh^{(j-1)}(\mu) & \mu^{N_f} (e^\mu \pm (-1)^{N_f} e^{-\mu}) & -\mu^{j-1} \sinh^{(j-1)}(\mu) \\ \mathbf{C}(\{\mu_i\}) & \mu_i^{N_f} (e^{\mu_i} \pm (-1)^{N_f} e^{-\mu_i}) & -\mathbf{S}(\{\mu_i\}) \\ \mathbf{C}(\{\mu_i\}) & \pm \mu_i^{N_f} (e^{\mu_i} \pm (-1)^{N_f} e^{-\mu_i}) & \mathbf{S}(\{\mu_i\}) \end{pmatrix} . \end{aligned} \quad (3.21)$$

We now have to treat each of the two cases \pm separately. For the “+” case the $(N_f + 1)$ -th column reads $(2\mu^{N_f} \cosh^{(N_f)}(\mu), 2\mu_1^{N_f} \cosh^{(N_f)}(\mu_1), \dots, 2\mu_1^{N_f} \cosh^{(N_f)}(\mu_1), \dots)$ which can be absorbed into the matrices \mathbf{C} . The “-” case yields $(2\mu^{N_f} \sinh^{(N_f)}(\mu), 2\mu_1^{N_f} \sinh^{(N_f)}(\mu_1), \dots, -2\mu_1^{N_f} \sinh^{(N_f)}(\mu_1), \dots)$, and we can absorb it into the matrices \mathbf{S} after changing the sign of the vector and permuting it to the very right. We will then add or subtract blocks again to produce a $\mathbf{0}_{N_f \times N_f}$ block on the right or left, respectively. In more detail we have for the “+” case

$$\begin{aligned} \det \mathbf{D}(\mu, \{\mu_i\}) + (-1)^{N_f} \det \mathbf{D}(-\mu, \{-\mu_i\}) &= \\ &= 2 \det \begin{pmatrix} \mu^{j-1} \cosh^{(j-1)}(\mu) & -\mu^{j-1} \sinh^{(j-1)}(\mu) \\ \mathbf{C}(\{\mu_i\})_{N_f \times N_f+1} & -\mathbf{S}(\{\mu_i\})_{N_f \times N_f} \\ \mathbf{C}(\{\mu_i\})_{N_f \times N_f+1} & \mathbf{S}(\{\mu_i\})_{N_f \times N_f} \end{pmatrix} \\ &= 2 \det \begin{pmatrix} \mu^{j-1} \cosh^{(j-1)}(\mu) & -\mu^{j-1} \sinh^{(j-1)}(\mu) \\ 2\mathbf{C}(\{\mu_i\})_{N_f \times N_f+1} & \mathbf{0}_{N_f \times N_f} \\ \mathbf{C}(\{\mu_i\})_{N_f \times N_f+1} & \mathbf{S}(\{\mu_i\})_{N_f \times N_f} \end{pmatrix} . \end{aligned} \quad (3.22)$$

Because of the zero block $\mathbf{0}_{N_f \times N_f}$, a Laplace expansion into products of $(N_f + 1) \times (N_f + 1)$ blocks and $N_f \times N_f$ blocks contains at most $N_f + 1$ nonzero terms. However, all the determinants for which two rows of the matrices \mathbf{C} occur twice, also vanish. Thus we are left with

$$\det \mathbf{D}(\mu, \{\mu_i\}) + (-1)^{N_f} \det \mathbf{D}(-\mu, \{-\mu_i\}) = 2^{N_f+1} \det \mathbf{C}(\mu, \{\mu_i\}) \det \mathbf{S}(\{\mu_i\}), \quad (3.23)$$

where the size of the determinants on the right hand side is given by the number of arguments.

Along the same lines we obtain for the “-”-case,

$$\begin{aligned} \det \mathbf{D}(\mu, \{\mu_i\}) - (-1)^{N_f} \det \mathbf{D}(-\mu, \{-\mu_i\}) &= \\ &= (-1)^{N_f+1} \det \begin{pmatrix} \mu^{j-1} \cosh^{(j-1)}(\mu) & -\mu^{j-1} \sinh^{(j-1)}(\mu) \\ \mathbf{C}(\{\mu_i\})_{N_f \times N_f} & -\mathbf{S}(\{\mu_i\})_{N_f \times N_f+1} \\ \mathbf{C}(\{\mu_i\})_{N_f \times N_f} & \mathbf{S}(\{\mu_i\})_{N_f \times N_f+1} \end{pmatrix} \\ &= (-1)^{N_f+1} \det \begin{pmatrix} \mu^{j-1} \cosh^{(j-1)}(\mu) & -\mu^{j-1} \sinh^{(j-1)}(\mu) \\ \mathbf{0}_{N_f \times N_f} & -2\mathbf{S}(\{\mu_i\})_{N_f \times N_f+1} \\ \mathbf{C}(\{\mu_i\})_{N_f \times N_f} & \mathbf{S}(\{\mu_i\})_{N_f \times N_f+1} \end{pmatrix} \\ &= 2^{N_f+1} \det \mathbf{C}(\{\mu_i\}) \det \mathbf{S}(\mu, \{\mu_i\}). \end{aligned} \quad (3.24)$$

In the last step the minus signs coming from permuting the $(N_f + 1)$ -th column of eq. (3.21) get absorbed into the sign of the only non-vanishing contribution from the Laplace expansion. Taking eqs. (3.23) and (3.24) together with eq. (3.19) we finally obtain

$$\mathcal{Z}_{\text{QCD}_3^\pm}^{(2N_f+1)}(\mu, \{\mu_i\}) = \frac{(-1)^{N_f} 2^{N_f}}{\Delta(\{\mu_i^2\})^2 \prod_{k=1}^{N_f} (\mu^2 - \mu_k^2) \mu_k} \begin{cases} \det \mathbf{C}(\mu, \{\mu_i\}) \det \mathbf{S}(\{\mu_i\}) & \text{for } + \\ \det \mathbf{S}(\mu, \{\mu_i\}) \det \mathbf{C}(\{\mu_i\}) & \text{for } - \end{cases}. \quad (3.25)$$

We can now compare to the right hand side of eq. (3.18) with the help of eq. (3.14). Looking at the factor $\prod_k (\mu_k^2 - \mu^2)$ of the Vandermonde determinant of $\mathcal{Z}_{\nu}^{(N_f+1)}(\mu, \{\mu_i\})$ we obtain an additional factor $(-1)^{N_f}$, which adds up to the correct prefactor of eq. (3.18).

The first two theorems stated so far relate the QCD₃ partition function with an odd or even number of flavors to a single product of two QCD₄ partition functions. In the following we wish to consider the QCD₃ partition function with additional flavors which do not come in pairs of opposite sign. However, such partition functions do not satisfy relations as simple as Theorem I and II. Instead, they are given by linear combinations of products of QCD₄ partition functions and do not directly give relations between the spectral density of QCD₃ and QCD₄. For this reason we will only treat the simplest case, that of the even flavor QCD₃ partition function with two additional unpaired masses.

THEOREM III - Additional unpaired flavors in QCD₃:

Let in addition $\mathcal{Z}_{\text{QCD}_3}^{(2N_f+2)}(\{\mu_i\}, \zeta, \omega)$ be defined as

$$\mathcal{Z}_{\text{QCD}_3}^{(2N_f+2)}(\{\mu_i\}, \zeta, \omega) = (-1)^{(N_f+1)(N_f+2)/2} \frac{\det \begin{pmatrix} \mathbf{A}(\{\mu_i\}, \zeta) & \mathbf{A}(\{-\mu_i\}, -\zeta) \\ \mathbf{A}(\{-\mu_i\}, \omega) & \mathbf{A}(\{\mu_i\}, -\omega) \end{pmatrix}}{\Delta(\mathcal{M})}, \quad (3.26)$$

where the $(N_f + 1) \times (N_f + 1)$ matrix $\mathbf{A}(\{\mu_i\}, \zeta)$ is defined as in eq. (3.2). The mass matrix is given by $\mathcal{M} = \text{diag}(\mu_1, \dots, \mu_{N_f}, \zeta, -\mu_1, \dots, -\mu_{N_f}, \omega)$. Then the two following relations hold:

$$\begin{aligned} \mathcal{Z}_{\text{QCD}_3}^{(2N_f+2)}(\{\mu_i\}, \pm \zeta, \omega) &= \pi^{N_f+1} \sqrt{\zeta \omega} \left[\mathcal{Z}_{\nu=-1/2}^{(N_f+2)}(\{\mu_i\}, \zeta, \omega) \mathcal{Z}_{\nu=+1/2}^{(N_f)}(\{\mu_i\}) \right. \\ &\quad \left. \mp \mathcal{Z}_{\nu=+1/2}^{(N_f+2)}(\{\mu_i\}, \zeta, \omega) \mathcal{Z}_{\nu=-1/2}^{(N_f)}(\{\mu_i\}) \right]. \end{aligned} \quad (3.27)$$

PROOF: In order to prove these two statements we will first rephrase them in terms of linear combinations of QCD₃ partition functions³:

$$\begin{aligned} \mathcal{Z}_{\text{QCD}_3}^{(2N_f+2)}(\{\mu_i\}, \zeta, \omega) &+ \mathcal{Z}_{\text{QCD}_3}^{(2N_f+2)}(\{\mu_i\}, -\zeta, \omega) = \\ &= 2\pi^{N_f+1} \sqrt{\zeta\omega} \mathcal{Z}_{\nu=-1/2}^{(N_f+2)}(\{\mu_i\}, \zeta, \omega) \mathcal{Z}_{\nu=+1/2}^{(N_f)}(\{\mu_i\}) , \end{aligned} \quad (3.28)$$

and

$$\begin{aligned} \mathcal{Z}_{\text{QCD}_3}^{(2N_f+2)}(\{\mu_i\}, \zeta, \omega) &- \mathcal{Z}_{\text{QCD}_3}^{(2N_f+2)}(\{\mu_i\}, -\zeta, \omega) = \\ &= -2\pi^{N_f+1} \sqrt{\zeta\omega} \mathcal{Z}_{\nu=+1/2}^{(N_f+2)}(\{\mu_i\}, \zeta, \omega) \mathcal{Z}_{\nu=-1/2}^{(N_f)}(\{\mu_i\}) . \end{aligned} \quad (3.29)$$

Clearly, when taking the sum or difference of these two equations we find back the theorem eq. (3.27). In the following we will reduce eqs. (3.28) to identities between determinants which will be then proven separately in Appendix A.

First of all, we note that, as in the proofs of the previous theorems on the QCD₃ side, we can again reorder the determinant in terms of hyperbolic functions. After permuting the additional arguments ζ and ω we obtain

$$\mathcal{Z}_{\text{QCD}_3}^{(2N_f+2)}(\{\mu_i\}, \zeta, \omega) = (-1)^{(N_f+1)(N_f+2)/2} 2^{N_f+1} \frac{\det \mathbf{D}(\{\mu_i\}, \zeta, \omega)}{\Delta(\{\mu_i\}, \{-\mu_i\}, \zeta, \omega)} , \quad (3.30)$$

where we define the $(2N_f + 2) \times (2N_f + 2)$ determinant \mathbf{D} for an even number of flavors,

$$\det \mathbf{D}(\{\mu_i\}, \zeta, \omega) = \det \begin{pmatrix} \mathbf{C}(\{\mu_i\}) & -\mathbf{S}(\{\mu_i\}) \\ \mathbf{C}(\{\mu_i\}) & \mathbf{S}(\{\mu_i\}) \\ \zeta^{j-1} \cosh^{(j-1)}(\zeta) & -\zeta^{j-1} \sinh^{(j-1)}(\zeta) \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & -\omega^{j-1} \sinh^{(j-1)}(\omega) \end{pmatrix} . \quad (3.31)$$

In the next step, we write out the Vandermonde determinants explicitly as in eq. (3.19),

$$\Delta(\{\mu_i\}, \{-\mu_i\}, \pm\zeta, \omega) = (-1)^{N_f(N_f+1)/2} (\omega \mp \zeta) \prod_{f=1}^{N_f} 2\mu_f \Delta(\{\mu_i^2\})^2 \prod_{j=1}^{N_f} (\omega^2 - \mu_j^2)(\zeta^2 - \mu_j^2) \quad (3.32)$$

for the QCD₃ side, and

$$\Delta(\{\mu_i^2\}, \zeta^2, \omega^2) \Delta(\{\mu_i^2\}) = (\omega^2 - \zeta^2) \Delta(\{\mu_i^2\})^2 \prod_{j=1}^{N_f} (\omega^2 - \mu_j^2)(\zeta^2 - \mu_j^2) \quad (3.33)$$

for the QCD₄ side. Putting everything together, we obtain from eq. (3.28)

$$(\omega + \zeta) \det \mathbf{D}(\{\mu_i\}, \zeta, \omega) + (\omega - \zeta) \det \mathbf{D}(\{\mu_i\}, -\zeta, \omega) = (-1)^{N_f+1} 2^{N_f+1} \det \mathbf{C}(\{\mu_i\}, \zeta, \omega) \det \mathbf{S}(\{\mu_i\}) \quad (3.34)$$

and from eq. (3.29)

$$(\omega + \zeta) \det \mathbf{D}(\{\mu_i\}, \zeta, \omega) - (\omega - \zeta) \det \mathbf{D}(\{\mu_i\}, -\zeta, \omega) = (-1)^{N_f} 2^{N_f+1} \det \mathbf{S}(\{\mu_i\}, \zeta, \omega) \det \mathbf{C}(\{\mu_i\}) . \quad (3.35)$$

These two relations between determinants are derived in Lemma 1 (A.1) and Lemma 2 (A.1) of Appendix A, which completes our proof of Theorem III.

³Note that there is nothing new in the definition (3.26) as compared with the original definition (3.1). In order to avoid confusion, we have indicated explicitly where the unpaired masses should enter in the individual blocks.

4 Applying the Replica Method: General Results

In this section we will use the replica method to derive general relations between the microscopic spectral densities of QCD₃ and QCD₄.

Let us first consider QCD₄ with N_f fermions in the fundamental representation. We add n valence fermions with degenerate (rescaled) mass μ_v to this theory. In such a theory, the resolvent, or partially quenched chiral condensate, is defined as

$$\frac{\Sigma_\nu(\mu_v, \{\mu_i\})}{\Sigma} \equiv \lim_{n \rightarrow 0} \frac{1}{n} \frac{\partial}{\partial \mu_v} \ln \mathcal{Z}_\nu^{(N_f+n)}(\mu_v, \{\mu_i\}) . \quad (4.36)$$

The partially quenched chiral condensate in QCD₃ is defined similarly (recalling that here the number of flavors is $2n$). The microscopic spectral density of the Dirac operator is given by the discontinuity of the resolvent across the imaginary axis [13],

$$\begin{aligned} \rho_S^{(\nu)}(\zeta; \{\mu_i\}) &= \frac{1}{2\pi} \text{Disc } \Sigma_\nu(\mu_v, \{\mu_i\})|_{\mu_v=i\zeta} \\ &= \frac{1}{2\pi} [\Sigma_\nu(i\zeta + \epsilon; \{\mu_i\}) - \Sigma_\nu(i\zeta - \epsilon; \{\mu_i\})] \end{aligned} \quad (4.37)$$

The analysis of Dirac operator spectra by means of a generating function of the resolvent is not restricted to just the spectral density. The k -point spectral correlation functions can be computed from partially quenched chiral susceptibilities $\chi(\mu_{v_1}, \dots, \mu_{v_k}, \{\mu_i\})$. These quantities can be derived from the partition function by extending it with kn valence fermions in k mass-degenerate sets. Then the $n \rightarrow 0$ limit is again taken after appropriate differentiations,

$$\chi_\nu^{(k)}(\mu_{v_1}, \dots, \mu_{v_k}, \{\mu_i\}) \equiv \lim_{n \rightarrow 0} \frac{1}{n^k} \frac{\partial}{\partial \mu_{v_1}} \dots \frac{\partial}{\partial \mu_{v_k}} \ln \mathcal{Z}_\nu^{(N_f+kn)}(\mu_{v_1}, \dots, \mu_{v_k}, \{\mu_i\}) . \quad (4.38)$$

In a spectral representation,

$$\chi_\nu^{(k)}(\mu_{v_1}, \dots, \mu_{v_k}, \{\mu_i\}) = \int_{-\infty}^{\infty} d\zeta_1 \dots d\zeta_k \frac{\rho_{S,\text{conn.}}^{(\nu)}(\zeta_1, \dots, \zeta_k; \{\mu_i\})}{(i\zeta_1 + \mu_{v_1}) \dots (i\zeta_k + \mu_{v_k})} , \quad (4.39)$$

and we can again invert this relation to get the spectral correlation function as the discontinuity across the imaginary mass axis,

$$\rho_{S,\text{conn.}}^{(\nu)}(\zeta_1, \dots, \zeta_k; \{\mu_i\}) = \frac{1}{(2\pi)^k} \text{Disc } \chi_\nu^{(k)}(\mu_{v_1}, \dots, \mu_{v_k}, \{\mu_i\}) \Big|_{\mu_{v_j}=i\zeta_j} . \quad (4.40)$$

For instance, for the 2-point function the explicit relation reads after using the symmetry of the spectrum around the origin

$$\begin{aligned} \rho_{S,\text{conn.}}^{(\nu)}(\zeta_1, \zeta_2; \{\mu_i\}) &= \frac{1}{4\pi^2} [\chi_\nu^{(2)}(i\zeta_1 + \epsilon, i\zeta_2 + \epsilon; \{\mu_i\}) + \chi_\nu^{(2)}(i\zeta_1 - \epsilon, i\zeta_2 - \epsilon; \{\mu_i\}) \\ &\quad - \chi_\nu^{(2)}(i\zeta_1 - \epsilon, i\zeta_2 + \epsilon; \{\mu_i\}) - \chi_\nu^{(2)}(i\zeta_1 + \epsilon, i\zeta_2 - \epsilon; \{\mu_i\})] . \end{aligned} \quad (4.41)$$

A similar formula gives us $\rho_{S,\text{conn.}}^{(\nu)}(\zeta_1, \dots, \zeta_k; \{\mu_i\})$ for arbitrary k .

4.1 The Microscopic Spectral Density

Let us first apply the relation (4.37) to the partition function identity (3.12), adding to the $2N_f$ physical fermions n valence fermions of mass μ_v and n valence fermions of mass $-\mu_v$. The number n is eventually taken to zero. We then obtain a relation between the microscopic spectral density of the Dirac operator in 2+1 dimensions for $2N_f$ fermions pairwise grouped with opposite mass and the corresponding microscopic spectral densities in QCD₄ *evaluated at half-integer topological charge*,

$$\begin{aligned}\rho_{\text{QCD}_3}^{(2N_f)}(\zeta; \{\mu_i\}) &= \frac{1}{2} \left[\rho_{\text{QCD}_3}^{(2N_f)}(\zeta; \{\mu_i\}) + \rho_{\text{QCD}_3}^{(2N_f)}(-\zeta; \{\mu_i\}) \right] \\ &= \frac{1}{2} \left[\rho_{\text{QCD}_4}^{(N_f, \nu=+1/2)}(\zeta; \{\mu_i\}) + \rho_{\text{QCD}_4}^{(N_f, \nu=-1/2)}(\zeta; \{\mu_i\}) \right].\end{aligned}\quad (4.42)$$

In the first equality we have used that the *average* spectral density is an even function of ζ . Note that the μ -independent factors in eqs. (3.12) and (3.18) are immaterial. The same relation can be derived from the Random Matrix Theory representation of the finite volume partition function (see Appendix B).

For $2N_f + 1$ flavors we only consider the case $\mathcal{Z}_{\text{QCD}_3^+}^{(2N_f+1)}$ for which the spectral density is positive definite. From the replica limit of the resolvent applied to (3.18) we find

$$\rho_{\text{QCD}_3^+}^{(2N_f+1)}(\zeta; \{\mu_i\}, \mu) = \frac{1}{2} \left[\rho_{\text{QCD}_4}^{(N_f, \nu=-1/2)}(\zeta; \{\mu_i\}, \mu) + \rho_{\text{QCD}_4}^{(N_f, \nu=+1/2)}(\zeta; \{\mu_i\}) \right], \quad (4.43)$$

where we again have used that the average spectral density is an even function of ζ . For $\mu = 0$ this relation can be simplified by applying flavor-topology duality [38],

$$\mathcal{Z}_{\text{QCD}_3^+}^{(2N_f+1)}(0, \{\mu_i\}) = \pi^{N_f} \prod_{j=1}^{N_f} \frac{1}{\mu_j} \left(\mathcal{Z}_{\nu=+1/2}^{(N_f)}(\{\mu_i\}) \right)^2. \quad (4.44)$$

Here, the prefactor gets modified compared to eq. (3.18) for the following reason. If we define all partition functions normalized according to eqs. (3.6) and (3.7) the flavor-topology shift for general ν (see eq. (3.11)) yields the following factor:

$$\lim_{\mu \rightarrow 0} \left(\frac{\pi\mu}{2} \right)^{1/2} \mathcal{Z}_{\nu=-1/2}^{(N_f+1)}(\mu, \{\mu_i\}) = \prod_{j=1}^{N_f} \frac{1}{\mu_j} \mathcal{Z}_{\nu=+1/2}^{(N_f)}(\{\mu_i\}). \quad (4.45)$$

Eq. (4.44) leads to

$$\rho_{\text{QCD}_3}^{(2N_f+1)}(\zeta; \{\mu_i\}, 0) = \rho_{\text{QCD}_4}^{(N_f, \nu=+1/2)}(\zeta; \{\mu_i\}), \quad (4.46)$$

a relation first discovered by Christiansen [7] in the context of Random Matrix Theory. Here we see that this relation follows from the effective field theory partition functions alone, after using the replica method. Note how the “topological” ν/μ -term (for $\nu = 1/2$) nicely has cancelled out on the right hand side.

The relations (4.42) and (4.43) are our two main results. They show that the microscopic spectral densities of QCD₃ follow directly from those of QCD₄ by analytic continuation in ν .

It is straightforward to check the relations (4.42) and (4.43) against the well-known expressions for the *massless* microscopic spectral densities for the two theories, as for example computed in Random Matrix Theory [4, 2],

$$\begin{aligned}
\rho_{\text{QCD}_3}^{(2N_f)}(\zeta; \{\mu_i = 0\}) &= \frac{\zeta}{4} \left[J_{N_f-1/2}(\zeta)^2 - J_{N_f+1/2}(\zeta) J_{N_f-3/2}(\zeta) \right. \\
&\quad \left. + J_{N_f+1/2}(\zeta)^2 - J_{N_f+3/2}(\zeta) J_{N_f-1/2}(\zeta) \right] , \\
\rho_{\text{QCD}_3}^{(2N_f+1)}(\zeta; \{\mu_i = 0\}, 0) &= \frac{\zeta}{2} \left[J_{N_f+1/2}(\zeta)^2 - J_{N_f+3/2}(\zeta) J_{N_f-1/2}(\zeta) \right] , \\
\rho_{\text{QCD}_4}^{(N_f, \nu)}(\zeta; \{\mu_i = 0\}) &= \frac{\zeta}{2} \left[J_{N_f+\nu}(\zeta)^2 - J_{N_f+\nu+1}(\zeta) J_{N_f+\nu-1}(\zeta) \right] . \tag{4.47}
\end{aligned}$$

The relations derived above are indeed seen to be satisfied.

It is also instructive to write down the asymptotic large mass expansion of the resolvent of the QCD₃ partition function from the known asymptotic expansion of the resolvent for the QCD₄ partition function. We only consider the case of an even number of massless flavors. The starting point is the relation (3.12) for $2N_f$ massless flavors and $2n$ valence flavors with masses in pairs $\pm\mu_v$. Using flavor-topology duality [38], the QCD₄ partition function can be rewritten using

$$\lim_{\mu \rightarrow 0} \prod_{k=1}^n \left[\Gamma(\nu + k) \left(\frac{2}{\mu} \right)^{\nu+k-1} \right] \mathcal{Z}_{\nu}^{(N_f+n)}(\{\mu, \dots, \mu, \mu_v\}) = \frac{\mathcal{Z}_{N_f+\nu}^{(n)}(\{\mu_v\})}{\mu_v^{n(\nu+N_f)}} \tag{4.48}$$

This relation is readily derived from the explicit form of the finite volume partition functions [38]. From (3.12) we then obtain

$$\mathcal{Z}_{\text{QCD}_3}^{(2N_f+2n)}(\{0, \dots, 0, \mu_v\}) = \frac{\pi^{N_f+n}}{2nN_f} \mathcal{Z}_{\nu=N_f-1/2}^{(n)}(\{\mu_v\}) \mathcal{Z}_{\nu=N_f+1/2}^{(n)}(\{\mu_v\}). \tag{4.49}$$

After taking the replica limit we find the relation

$$\frac{\Sigma_{\text{QCD}_3}^{2N_f}(\mu_v)}{\Sigma} = \frac{1}{2} \left[\frac{\Sigma_{\nu=N_f-1/2}^{N_f=0}(\mu_v)}{\Sigma} + \frac{\Sigma_{\nu=N_f+1/2}^{N_f=0}(\mu_v)}{\Sigma} \right] - \frac{N_f}{\mu_v}. \tag{4.50}$$

The replica limit of the quenched QCD₄ chiral condensate is given by (for $-\frac{3}{2}\pi < \arg \mu_v < \frac{\pi}{2}$) [31]

$$\begin{aligned}
\frac{\Sigma_{\nu}(\mu_v)}{\Sigma} &= 1 - \frac{i(-1)^{\nu} e^{-2\mu_v}}{2\mu_v} + \frac{(4\nu^2 - 1)(1 - i(-1)^{\nu} e^{-2\mu_v})}{8\mu_v^2} - \frac{i(-1)^{\nu} e^{-2\mu_v} (4\nu^2 - 1)(4\nu^2 - 9)}{64\mu_v^3} \\
&\quad + \frac{(4\nu^2 - 1)(4\nu^2 - 9) [i(-1)^{\nu} e^{-2\mu_v} (19 - 4\nu^2) - 6]}{3 \times 256\mu_v^4} + \dots . \tag{4.51}
\end{aligned}$$

We thus find the following large mass expansion for the resolvent of QCD₃

$$\begin{aligned}
\frac{\Sigma_{\text{QCD}_3}^{2N_f}(\mu_v)}{\Sigma} &= 1 - \frac{N_f}{\mu_v} + \frac{N_f^2}{2\mu_v^2} - \frac{N_f(N_f^2 - 1)}{4\mu_v^4} \\
&\quad + e^{-2\mu_v} (-1)^{N_f} \left[\frac{N_f}{2\mu_v^2} + \frac{N_f(N_f^2 - 1)}{2\mu_v^3} + \frac{N_f(N_f^2 - 1)(N_f^2 - 3)}{4\mu_v^4} \right] + \dots . \tag{4.52}
\end{aligned}$$

For $N_f = 1$ and $N_f = 2$ all higher-order terms vanish. A replica calculation thus provides us with the exact non-perturbative result.

4.2 Higher Order Correlation Functions

Let us now apply the relations between partition functions in QCD_3 and QCD_4 to higher order correlation functions. Because the valence masses occur in pairs $\pm\mu_k$ in the QCD_3 partition function, differentiation with respect to the valence quark masses does not exactly result in the k -point spectral correlation functions. Rather we obtain nontrivial relations between correlation functions with arguments that differ by a minus sign. As an example, let us look in more detail at the two-point correlation function. We find

$$\begin{aligned} & \rho_{\text{QCD}_3, \text{conn.}}^{(2N_f)}(\zeta_1, \zeta_2; \{\mu_i\}) + \rho_{\text{QCD}_3, \text{conn.}}^{(2N_f)}(\zeta_1, -\zeta_2; \{\mu_i\}) \\ + & \rho_{\text{QCD}_3, \text{conn.}}^{(2N_f)}(-\zeta_1, \zeta_2; \{\mu_i\}) + \rho_{\text{QCD}_3, \text{conn.}}^{(2N_f)}(-\zeta_1, -\zeta_2; \{\mu_i\}) \\ = & \rho_{S, \text{conn.}}^{(\nu=+1/2)}(\zeta_1, \zeta_2; \{\mu_i\}) + \rho_{S, \text{conn.}}^{(\nu=-1/2)}(\zeta_1, \zeta_2; \{\mu_i\}) . \end{aligned} \quad (4.53)$$

With connected two-point correlators given by the square of the kernel,

$$\begin{aligned} \rho_{\text{QCD}_3, \text{conn.}}^{(2N_f)}(\zeta_1, \zeta_2; \{\mu_i\}) &= - \left[K_{\text{UE}}^{(2N_f)}(\{\mu_i\}, \zeta_1, \zeta_2) \right]^2 , \\ \rho_{S, \text{conn.}}^{(\nu)}(\zeta_1, \zeta_2; \{\mu_i\}) &= - \left[K_{\text{chUE}}^{(N_f, \nu)}(\{\mu_i\}, \zeta_1^2, \zeta_2^2) \right]^2 , \end{aligned} \quad (4.54)$$

the relation (4.53) is in complete agreement with result from Random Matrix Theory obtained in Appendix B (see eq. (B.10)). Similar relations can be derived for higher order correlation functions.

We stress that the above relations are totally general, independent of any detailed evaluation of the $n \rightarrow 0$ limit of the partition functions involved. The replica method thus provides us with truly non-perturbative relations which go beyond asymptotic expansions. Actually, in this case the asymptotic expansions terminate, as we shall see below.

Finally, a note of caution for using the replica method. In our proofs we have not excluded the possible appearance of terms that vanish for positive integer values of n but are finite for $n \rightarrow 0$ (for example, terms of the form $\sin n\pi/n$). Because of the agreement with the exact results we know that such terms are absent. It is an interesting problem to understand why this is the case for an analytical continuation in the number of flavors whereas such terms appear, as we have seen in eq. (3.9), when we perform an analytical continuation in the topological charge.

5 Virasoro Constraints and Spectral Sum Rules

In the next section we will turn to explicit calculations based on the replica method. So far there are no known techniques that have succeeded in taking the replica limit $n \rightarrow 0$ beyond series expansions. The most obvious expansion that comes to mind is a Taylor series in the mass. Such an expansion was recently considered in the context of QCD_4 [30], where it was found that the so-called de Wit–’t Hooft poles prohibit the evaluation of the partially quenched chiral condensate $\Sigma_\nu(\mu_v, \{\mu_i\})$ beyond a given order, depending on the value of N_f and ν . In QCD_4 , these de Wit–’t Hooft poles occur at precisely the order for which the logarithmic terms appear in the expansion [30, 31]. As a particular consequence, the small-mass expansion cannot be used to derive spectral correlation functions. There is simply no discontinuity across the imaginary mass axis up to the order at which the replica method predicts the partially quenched chiral susceptibilities.⁴

⁴Resummations of the series may hold a clue to how this obstacle can be overcome; see section 4.2 of ref. [31].

As a preparation for the next section we derive in this section the small-mass expansions of the partition functions (2.7) and (2.10). Taking advantage of the results obtained we also derive the corresponding spectral sum rules, thus generalizing previous results. The small-mass expansions of (2.7) and (2.10) can be obtained by several different methods such as the character expansion of [36]⁵. We can also use the known results for the effective partition function of QCD₄ as a generator for unitary integrals (as the effective QCD₄ partition function is simply an external field integral over the group U(N_f)). However, the most elegant and efficient way to push the small mass expansion to arbitrarily high orders is probably through the use of Virasoro constraints. The appearance of Virasoro constraints in unitary integrals was first noticed in [41] for the case of large masses. It was later generalized for the case of small masses in [42]. In both works, only the one-link integral (2.12) at zero topological charge ($\nu = 0$) was studied, although in the small-mass expansion the flavor-topology duality readily gives the analytical continuation to $\nu \neq 0$.

For the derivation of Virasoro constraints for QCD₃ it is convenient to start with the following more general partition function:

$$\mathcal{Z}_q^{(N)}(M) = \int dU \exp \left[\text{Tr} M U I_q U^\dagger \right] , \quad (5.1)$$

where M is an arbitrary $N \times N$ matrix and I_q is such that $I_q^2 = \mathbf{1}_{N \times N}$ and $\text{Tr} I_q = q$. In particular, the even-flavor partition function of eq. (2.7) is obviously of this kind (with $q = 0$ and $M = V \Sigma M$), and it turns out that in the small-mass expansion other detailed properties of I_q are irrelevant. Thus, in the small-mass expansion the partition functions (5.1) with $q = 0, \pm 1$ are the building blocks for the QCD₃ partition functions given by (2.7), (2.10) and (2.11). It was noted in ref. [30] that the finite-volume effective field theory partition functions satisfy the same type of differential equation, whether in 3 or 4 dimensions. Indeed, from the properties of I_q we find

$$\left(\frac{\partial^2}{\partial M_{ca} \partial M_{ab}} - \delta_{cb} \right) \mathcal{Z}_q^{(N)} = 0 \quad (5.2)$$

$$\delta^{ab} \frac{\partial \mathcal{Z}_q^{(N)}}{\partial M_{ab}} = q \mathcal{Z}_q^{(N)}. \quad (5.3)$$

Due to the unitary invariance of the measure it is clear that the partition function, (5.1), depends only on the traces of the matrix M . Thus, following [42], we can change variables from the matrix elements M_{ab} to

$$t_k = \frac{1}{k} \text{Tr} M^k \quad , \quad k \geq 1. \quad (5.4)$$

Clearly, for finite N not all t_k are independent. Using the chain rule, the differential equations (5.2) and (5.3) can be rewritten as

$$\sum_{s \geq 2} M_{bc}^{s-2} [\mathcal{L}_s - \delta_{s,2}] \mathcal{Z}_q^{(N)} = 0, \quad (5.5)$$

$$(\mathcal{L}_1 - q) \mathcal{Z}_q^{(N)} = 0, \quad (5.6)$$

where the operators \mathcal{L}_s are defined by

$$\mathcal{L}_s = \sum_{j=1}^{s-1} \partial_j \partial_{s-j} + \sum_{j \geq 1} j t_j \partial_{j+s} + N \partial_s, \quad s \geq 1. \quad (5.7)$$

⁵We have explicitly verified that the more general expansion we shall provide below agrees up to fourth order with the character expansion of [36] for the case $\text{Tr}(\mathcal{M}) = 0$.

We use the notation $\partial_j = \partial/\partial t_j$, and our convention is that for $s = 1$ the first term in \mathcal{L}_s is absent. The \mathcal{L}_s satisfy the Virasoro algebra⁶

$$[\mathcal{L}_r, \mathcal{L}_s] = (r - s)\mathcal{L}_{r+s}. \quad (5.8)$$

A sufficient condition for the solution of the differential equations (5.2) and (5.3) is thus given by so-called Virasoro constraints:

$$\mathcal{L}_s \mathcal{Z}_q^{(N)} = (\delta_{s,2} + q\delta_{s,1})\mathcal{Z}_q^{(N)}, \quad s \geq 1. \quad (5.9)$$

They differ from the Virasoro constraints for QCD₄ by the presence of the term $\delta_{s,2}$ and the parameter q which takes the value $q = 1$ in the QCD₄ case.

We normalize the partition function by $\mathcal{Z}_q^{(N)}(t_k = 0) = 1$. The Virasoro constraints can then be solved iteratively order by order in the masses by letting $t_k = 0$ after acting with the differential operators. It is instructive to consider the solution of the lowest few orders. Using the Ansatz,

$$\mathcal{Z}_q^{(N)} = 1 + at_1 + b_1 t_1^2 + b_2 t_2 + \mathcal{O}(\mu^3), \quad (5.10)$$

we deduce from $\mathcal{L}_1 \mathcal{Z}_q^{(N)} = q\mathcal{Z}_q^{(N)}$ at $t_k = 0$,

$$Na = q. \quad (5.11)$$

At order μ^2 we can use $\partial_1 \mathcal{L}_1 \mathcal{Z}_q^{(N)} = q\partial_1 \mathcal{Z}_q^{(N)}$ and $\mathcal{L}_2 \mathcal{Z}_q^{(N)} = \mathcal{Z}_q^{(N)}$ at $t_k = 0$ to obtain the equations,

$$\begin{aligned} 2Nb_1 + b_2 &= qa = \frac{q^2}{N}, \\ 2b_1 + Nb_2 &= 1. \end{aligned} \quad (5.12)$$

The solution is given by $b_2 = (N^2 - q^2)/(N(N^2 - 1))$ and $b_1 = (q^2 - 1)/(2(N^2 - 1))$. Proceeding like this, we have derived the small-mass expansion of $\mathcal{Z}_q^{(N)}$ up to order μ^6 , and the expansion can easily be pushed higher. We quote here the $\mathcal{O}(\mu^4)$ result, which is given by

$$\begin{aligned} \mathcal{Z}_q^{(N)} &= 1 + \frac{q}{N} \text{Tr} M + \frac{(N^2 - q^2) \text{Tr} M^2}{2N(N^2 - 1)} + \frac{(q^2 - 1)(\text{Tr} M)^2}{2(N^2 - 1)} - \frac{2q(N^2 - q^2)}{3N(N^2 - 1)(N^2 - 4)} \text{Tr} M^3 \\ &+ \frac{q(N^2 - q^2)}{2(N^2 - 1)(N^2 - 4)} \text{Tr} M \text{Tr} M^2 + \frac{q}{6N} \left[\frac{N^2(q^2 - 3) + 4 - 2q^2}{(N^2 - 1)(N^2 - 4)} \right] (\text{Tr} M)^3 \\ &+ \frac{(N^2 - q^2)(5q^2 + 4 - N^2)}{4N(N^2 - 1)(N^2 - 4)(N^2 - 9)} \text{Tr} M^4 + \frac{(N^2 - q^2)(3N^2 + 3q^2 - 2N^2 q^2 - 12)}{3N^2(N^2 - 1)(N^2 - 4)(N^2 - 9)} \text{Tr} M \text{Tr} M^3 \\ &+ \frac{(N^2 - q^2)(24 - 10N^2 + N^4 - 6q^2 - N^2 q^2)}{8N^2(N^2 - 1)(N^2 - 4)(N^2 - 9)} (\text{Tr} M^2)^2 + \frac{(N^2 - q^2)(q^2 - 1)}{4N(N^2 - 1)(N^2 - 9)} (\text{Tr} M)^2 \text{Tr} M^2 \\ &+ \frac{q^2 [N^4(q^2 - 6) - 8N^2(q^2 - 5) + 6(q^2 - 4)] + 3N^2(N^2 - 4)}{24N^2(N^2 - 1)(N^2 - 4)(N^2 - 9)} (\text{Tr} M)^4 + \mathcal{O}(\mu^5). \end{aligned} \quad (5.13)$$

In Appendix C we give the expansion up to order μ^6 . At order μ^5 we have found simple poles⁷ at $N^2 = 0, 1, 4, 9$ and 16 , while at order μ^6 , besides the simple poles at $N^2 = 0, 1, 4, 9, 16$ and 25 , a double pole appears at $N = 1$, in agreement with the calculation of unitary integrals in reference [43]. We can

⁶A “classical” Virasoro algebra, since there is no central charge. It is also known as a Witt algebra.

⁷We ignore the poles at unphysical negative values of N , which are completely irrelevant here.

check that our small-mass expansion of eqs. (5.13), (C.1) and (C.2) is correct by considering several special cases. For instance, if we take the mass matrix proportional to the identity $M = m \mathbf{1}_{N \times N}$ all poles cancel, and, according to our normalization, we should obtain $\mathcal{Z}_q = e^{mq}$. Its small-mass expansion is indeed in agreement with our results. This is one non-trivial check, for an arbitrary number of flavors and arbitrary value of q . We get another simple but non-trivial check for different masses by comparing our result with the small-mass expansion of the exact result [36] for the partition function $\mathcal{Z}_q^{(N)}$ at fixed value of the number of flavors N . For example, for $N = 2$ with mass matrix $M = \text{diag}(\mu_1, \mu_2)$ and $q = 0$, the partition function is given by

$$\mathcal{Z}_{\text{QCD}_3}^{(2)}(\mu_1, \mu_2) = \frac{\sinh(\mu_1 - \mu_2)}{\mu_1 - \mu_2}. \quad (5.14)$$

Expanding this expression for small masses, and comparing with eq. (5.13) and the corresponding expressions in Appendix C, we find perfect agreement.

It is important to realize that all the poles at integer values of N *cancel exactly* once we specify whether N is even ($q = 0$) or odd ($q = 1$). Thus neither the even- N nor the odd- N partition functions have any physical poles at all.

We have also checked that the expansion (5.13) is consistent with the expansion obtained from eq. (3.12) and the known small-mass expansion of the QCD₄ partition function from ref. [30] for $\text{Tr}(\mathcal{M}) = 0$. It is instructive to follow the way the coefficients of the expansion of the QCD₄ partition functions at $\nu = \pm 1/2$ combine to yield the corresponding coefficients for QCD₃ as in eq. (5.13). One observes that the appearance of the $\nu = \pm 1/2$ factor on the right hand side of eq. (3.12) is related to the *doubling* in number of flavors on the left hand side. As a simple example, let us consider

$$\mathcal{Z}_\nu^{(N_f)}(\mu) = \mu^{N_f \nu} \left(1 + \frac{N_f \mu^2}{4(N_f + \nu)} + \dots \right), \quad (5.15)$$

for N_f equal-mass flavors. This gives

$$\begin{aligned} \mathcal{Z}_{\nu=1/2}^{(N_f)}(\mu) \mathcal{Z}_{\nu=-1/2}^{(N_f)}(\mu) &= 1 + \frac{N_f}{2} \mu^2 \left(\frac{1}{2N_f + 1} + \frac{1}{2N_f - 1} \right) + \dots \\ &= 1 + \frac{1}{2} \frac{(2N_f)^2}{(2N_f)^2 - 1} \mu^2 + \dots, \end{aligned} \quad (5.16)$$

which coincides with our expansion (5.13) to this order (in agreement with equation (3.12) in the present normalization, where $N = 2N_f$ and $q = 0$).

For an *even* number of eigenvalues of the Dirac operator the effective finite-volume partition functions for QCD₃, eqs. (2.7) and (2.10), yield the same small-mass expansions as $\mathcal{Z}_{\text{QCD}_3}^{(N)}(\mathcal{M}) = (1/2)(\mathcal{Z}_q^{(N)}(\mathcal{M}V\Sigma) + \mathcal{Z}_{-q}^{(N)}(\mathcal{M}V\Sigma))$ where $q = 0, 1$ for an even and odd number of flavors, respectively. Comparing the expansion (5.13) with the small mass expansion of the QCD₃ partition function, $\langle \prod_{a=1}^N \det(D + m_a) \rangle_{\text{Yang-Mills}}$, we can read off the massless spectral sum rules for the QCD₃ Dirac operator

$$\begin{aligned}
\left\langle \sum_n \frac{1}{\zeta_n^2} \right\rangle &= \frac{N^2 - q^2}{N(N^2 - 1)}, \\
\left\langle \left(\sum_n \frac{1}{\zeta_n} \right)^2 \right\rangle &= \frac{1 - q^2}{N^2 - 1}, \\
\left\langle \sum_n \frac{1}{\zeta_n^4} \right\rangle &= \frac{(N^2 - q^2)(N^2 - 5q^2 - 4)}{N(N^2 - 1)(N^2 - 4)(N^2 - 9)}, \\
\left\langle \sum_n \frac{1}{\zeta_n^2} \left(\sum_m \frac{1}{\zeta_m} \right)^2 \right\rangle &= \frac{(N^2 - q^2)(1 - q^2)}{N(N^2 - 1)(N^2 - 9)}, \\
\left\langle \left(\sum_n \frac{1}{\zeta_n} \right)^4 \right\rangle &= \frac{q^2 [N^4(q^2 - 6) - 8N^2(q^2 - 5) + 6(q^2 - 4)] + 3N^2(N^2 - 4)}{N^2(N^2 - 1)(N^2 - 4)(N^2 - 9)}, \\
\left\langle \sum_n \frac{1}{\zeta_n} \sum_m \frac{1}{\zeta_m^3} \right\rangle &= \frac{(N^2 - q^2)(3N^2 + 3q^2 - 2N^2q^2 - 12)}{N^2(N^2 - 1)(N^2 - 4)(N^2 - 9)}, \\
\left\langle \left(\sum_n \frac{1}{\zeta_n^2} \right)^2 \right\rangle &= \frac{(N^2 - q^2)(24 - 10N^2 + N^4 - 6q^2 - N^2q^2)}{N^2(N^2 - 1)(N^2 - 4)(N^2 - 9)}. \tag{5.17}
\end{aligned}$$

The sum rules obtained at order μ^6 are given in the Appendix B. It should be noticed that some sum rules given in (5.17) and (C.4) could not be derived had we assumed $\text{Tr} M^{2k+1} = 0$ from the start. The point is that we only deduce sum rules for the massless theory. The quark masses only act as sources which should be as general as possible; otherwise some sum rules cannot be derived due to the vanishing of the corresponding source term.

The first two massless sum rules agree with what was found in the original paper [2] for even and odd flavors. We have also explicitly checked that the “diagonal” sum rules in (5.17) and (C.4) agree with the results obtained from Random Matrix Theory by integrating the first two formulas of (4.47). For the higher point functions a direct comparison with higher spectral correlation functions obtained from Random matrix theory becomes quite cumbersome. We have instead generated general relations among sum rules directly from the corresponding Random Matrix Theory via Schwinger-Dyson equations. For instance, from the identity

$$\int_{-\infty}^{\infty} \prod_{j=1}^{\mathcal{N}} d\lambda_j \sum_l \partial_l \left[\frac{1}{\lambda_l} \left(\sum_k \frac{1}{\lambda_k} \right)^a \Delta^2(\{\lambda_i\}) \prod_{p=1}^{\mathcal{N}} \left(\lambda_p^{2N_f} e^{-\frac{\mathcal{N}\Sigma^2}{2} \lambda_p^2} \right) \right] = 0, \tag{5.18}$$

we derive, after the rescaling $\zeta_k = \mathcal{N}\Sigma\lambda_k$,

$$\left\langle \left(\sum_k \frac{1}{\zeta_k} \right)^{a+2} \right\rangle + a \left\langle \sum_k \frac{1}{\zeta_k^3} \left(\sum_l \frac{1}{\zeta_l} \right)^{a-1} \right\rangle + \left\langle \left(\sum_k \frac{1}{\zeta_k} \right)^a \right\rangle = 2N_f \left\langle \sum_k \frac{1}{\zeta_k^2} \left(\sum_l \frac{1}{\zeta_l} \right)^a \right\rangle. \tag{5.19}$$

We have checked that relation (5.19) for $a = 0, 2, 4$, as well as other similarly derived relations, are in agreement with (5.17) and (C.4).

For even \mathcal{N} , the spectral sum rules that are odd under $\zeta_k \rightarrow -\zeta_k$ vanish identically as in the case of, *e.g.*, $\langle \sum_k (1/\zeta_k) \rangle = 0$. This is a consequence of the symmetry $\mathcal{Z}_{\text{QCD}_3}^{(N)}(-M) = \mathcal{Z}_{\text{QCD}_3}^{(N)}(M)$. This symmetry is clear for (2.10), and for (2.7) it follows from the following argument. In a small-mass expansion of (2.10) for each power of the matrix M we have the same power of the matrix Γ_5 and the trace of odd powers of the mass matrix must be accompanied by the trace of odd powers of Γ_5 which vanishes identically.

6 The Replica Limit of the QCD₃ Finite Volume Partition Function

In this section, starting from the replica limit of the small and large-mass expansions of the QCD₃ finite volume partition functions, we obtain the corresponding expansions of the partially quenched chiral condensate.

6.1 Small-Mass Expansion

Once we have the expansion of the QCD₃ partition function, it should in principle be a simple matter to use the replica method to derive the partially quenched chiral condensate. However, in the case of the small-mass expansion this is not at all the case. The reason is the apparent poles in the expansion. As mentioned above, whether N is even or odd, all these poles cancel out exactly in the expansion of the partition function itself. But with the replica method we are forced to consider non-integer values of $N = 2N_f + 2n$, where n is taken to zero. Then these so-called de Wit-'t Hooft poles [45] are significant: the analytic continuation cannot be performed across such poles. For larger and larger values of N_f the first pole is pushed to higher and higher order, but there will always be a finite order beyond which we cannot construct $\Sigma(\mu_v; \{\mu_i\})$ from a small-mass expansion using the replica method. This issue was discussed at length in the case of QCD₄ [30], and we shall therefore be brief here.

For simplicity, let us restrict ourselves to the case of massless physical fermions; the general case can be derived in a completely analogous manner. For a mass matrix \mathcal{M} with $2N_f$ massless flavors and n replica flavors of mass $+\mu_v$ paired with n replicas of mass $-\mu_v$ we have $N = 2n + 2N_f$ and,

$$\begin{aligned} \text{Tr}(\mathcal{M}^k) &= 0, \quad \text{for odd } k, \\ \text{Tr}(\mathcal{M}^k) &= 2n\mu_v^k, \quad \text{for even } k. \end{aligned} \quad (6.1)$$

Using (4.36), we formally find from the expansion of the partition function (5.13) and (C.2):

$$\begin{aligned} \frac{\Sigma(\mu_v)}{\Sigma} &= \frac{(4N_f^2 - q^2)}{2N_f(4N_f^2 - 1)} \mu_v + \frac{(4N_f^2 - q^2)(5q^2 + 4 - 4N_f^2)}{2N_f(4N_f^2 - 1)(4N_f^2 - 4)(4N_f^2 - 9)} \mu_v^3 \\ &+ \frac{2(4N_f^2 - q^2) \left[(4N_f^2 - 4)(4N_f^2 - 16) + 21q^4 - 14q^2 4N_f^2 + 140q^2 \right]}{2N_f(4N_f^2 - 1)(4N_f^2 - 4)(4N_f^2 - 9)(4N_f^2 - 16)(4N_f^2 - 25)} \mu_v^5 + \dots \quad (6.2) \end{aligned}$$

Here we have $q = 0$ for integer N_f and $q = 1$ for half-integer N_f , and we always consider the case of an even number of Dirac eigenvalues. As stressed above, the above expansion is only meaningful up to the order at which a pole has to be crossed in order to take the replica limit. The result (6.2) can now be compared with the expansion of the result obtained from Random Matrix Theory [2]⁸:

$$\begin{aligned} \frac{\Sigma_{\text{RMT}}(\mu_v)}{\Sigma}(2N_f = 1) &= \mu_v - \frac{2}{3}\mu_v^2 + \frac{1}{3}\mu_v^3 - \frac{2}{15}\mu_v^4 + \frac{2}{45}\mu_v^5 + \dots, \\ \frac{\Sigma_{\text{RMT}}(\mu_v)}{\Sigma}(2N_f = 2) &= \frac{2}{3}\mu_v - \frac{1}{3}\mu_v^2 + \frac{2}{15}\mu_v^3 - \frac{2}{45}\mu_v^4 + \frac{4}{315}\mu_v^5 + \dots, \\ \frac{\Sigma_{\text{RMT}}(\mu_v)}{\Sigma}(2N_f = 3) &= \frac{1}{3}\mu_v - \frac{1}{15}\mu_v^3 + \frac{2}{45}\mu_v^4 - \frac{2}{105}\mu_v^5 + \dots, \\ \frac{\Sigma_{\text{RMT}}(\mu_v)}{\Sigma}(2N_f = 4) &= \frac{4}{15}\mu_v - \frac{4}{105}\mu_v^3 + \frac{1}{45}\mu_v^4 - \frac{8}{945}\mu_v^5 + \dots, \end{aligned}$$

⁸For positive replica mass. The condensate is odd in the mass, and appropriate absolute value signs have to be inserted for $\mu_v < 0$; for example $\mu_v^2 \rightarrow \mu_v |\mu_v|$, etc..

$$\frac{\Sigma_{\text{RMT}}(\mu_v)}{\Sigma}(2N_f = 5) = \frac{1}{5}\mu_v - \frac{1}{105}\mu_v^3 + \frac{2}{945}\mu_v^5 + \dots \quad (6.3)$$

Comparing with the expression (6.2) obtained from the replica method we indeed find exact agreement precisely up to the order at which the replica method no longer can be applied, as a de Wit–t Hooft pole has to be crossed. We nevertheless notice that, remarkably, *all* odd-order coefficients agree with those of Random Matrix Theory if one proceeds beyond the order at which the replica method ceases to be valid. This is perhaps less mysterious if one considers the fact that as $N_f \rightarrow \infty$ the replica expansion becomes exact to all orders (and this exact expansion has just odd powers). Because the replica method applied to the small-mass expansion only gives us the (correct) series up to the first de Wit–t Hooft pole, we can only extract information about the microscopic spectral density of the Dirac operator up to that power in the mass which is free from such poles.

6.2 Large-Mass Expansion

In this subsection we use the replica method to calculate the large-mass expansion of the partially quenched chiral condensate. Because the expansion terminates, we are able to derive the exact microscopic spectral density of the Dirac operator in QCD₃ *without* relying on the assumption that it is the universal microscopic spectral density of Random Matrix Theory. We perform this calculation for one and two massless flavors by a direct replica calculation of the partition function (2.7).

We evaluate the partition function (2.7) with U an $2(N_f + n) \times 2(N_f + n)$ unitary matrix. The mass matrix is taken to be diagonal with elements $M_{k+N_f+n} = -M_k$. Ultimately, we are interested in the case $M_1 = \dots = M_{N_f} = 0$, and the remaining replica masses equal to $\pm\mu$. The integral (2.7) can be rewritten as [25],

$$\mathcal{Z}_{\text{QCD}_3}^{(2N_f+2n)} = \int_{-1}^1 d\lambda_1 \dots \int_{-1}^1 d\lambda_{N_f+n} dU_1 dU_2 \Delta^2(\Lambda) e^{\text{Tr} M_+ (U_1 \Lambda U_1^\dagger + U_2 \Lambda U_2^\dagger)}, \quad (6.4)$$

where U_1 and U_2 are $(N_f + n) \times (N_f + n)$ unitary matrices, M_+ is a diagonal matrix with diagonal elements given by $M_{+kk} = M_{kk}$. The diagonal matrix Λ contains the integration variables as diagonal entries. The Vandermonde determinant is denoted by $\Delta(\Lambda) = \prod_{k>l} (\lambda_k - \lambda_l)$. The integrals over U_1 and U_2 are familiar Itzykson-Zuber integrals. The partition function is thus given by

$$\mathcal{Z}_{\text{QCD}_3}^{(2N_f+2n)} = \int_{-1}^1 d\lambda_1 \dots \int_{-1}^1 d\lambda_{N_f+n} \frac{\det^2 e^{M_k \lambda_l}}{\Delta^2(M)}. \quad (6.5)$$

In the limit of N_f massless flavors and n quark masses equal to μ , the determinant can be rewritten as

$$\frac{\det e^{M_k \lambda_l}}{\Delta(M)} = \frac{1}{\mu^{N_f n} \prod_{k=1}^{N_f-1} k! \prod_{l=1}^{n-1} l!} \begin{vmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_{N_f+n} \\ \vdots & & \vdots \\ \lambda_1^{N_f-1} & \dots & \lambda_{N_f+n}^{N_f-1} \\ e^{\mu \lambda_1} & \dots & e^{\mu \lambda_{N_f+n}} \\ \lambda_1 e^{\mu \lambda_1} & \dots & \lambda_{N_f+n} e^{\mu \lambda_{N_f+n}} \\ \vdots & & \vdots \\ \lambda_1^{n-1} e^{\mu \lambda_1} & \dots & \lambda_{N_f+n}^{n-1} e^{\mu \lambda_{N_f+n}} \end{vmatrix}. \quad (6.6)$$

In general, this expression is rather complicated. However, the leading term in an expansion in $1/\mu$ can be obtained easily. We will first derive this term and then we will calculate the complete expansion for $N_f = 1$ and $N_f = 2$.

There are two types of integrations: those that are accompanied by an exponential $\exp \mu \lambda_k$, and those that are not. The second type of integrals correspond to zero modes in a saddle point expansion and have to be evaluated exactly. The remaining integrals are evaluated perturbatively by writing

$$\int_{-1}^1 d\lambda_k \cdots = \int_{-\infty}^1 d\lambda_k \cdots - \int_{-\infty}^{-1} d\lambda_k \cdots. \quad (6.7)$$

and putting $\lambda_k = 1 - x_k/\mu$ in the integrals over the segment $\langle -\infty, 1]$, and $\lambda_k = -1 - x_k/\mu$ in the integrals over the segment $\langle -\infty, -1]$. In both cases the integral over the x_k ranges over the segment $[0, \infty)$. Denoting by p the number of integrals with -1 as upper limit, the leading order contribution is obtained by keeping only the $p = 0$ terms. The leading order term is then given by the diagonal term in the expansion of the square of the determinant. All terms are the same and we thus find

$$\begin{aligned} \mathcal{Z}_{\text{QCD}_3}^{(2N_f+2n)} = & \frac{e^{2\mu n}}{\mu^{n^2+2N_f n}} \int_{-1}^1 d\lambda_1 \cdots \int_{-1}^1 d\lambda_{N_f} \Delta^2(\lambda_1, \dots, \lambda_{N_f}) \\ & \times \int_0^\infty dx_{N_f+1} \cdots \int_0^\infty dx_{N_f+n} e^{-2x_{N_f+1} - \cdots - 2x_{N_f+n}} \Delta^2(x_{N_f+1}, \dots, x_{N_f+n}). \end{aligned} \quad (6.8)$$

We recognize the partition function for the Laguerre ensemble which can easily be evaluated by means of the orthogonal polynomial method. However, we do not need this constant to find its leading order large- μ contribution to the resolvent. In the replica limit $n \rightarrow 0$ it is simply given by

$$\mathcal{Z}_{\text{QCD}_3}^{(2N_f+2n,p=0)} = \frac{e^{2n\mu}}{\mu^{2nN_f}}, \quad (6.9)$$

so that the μ -dependence of the resolvent becomes

$$\Sigma(\mu) = 1 - \frac{2N_f}{\mu} + O(1/\mu^2). \quad (6.10)$$

Let us return again to the group integral (2.7). For $p \neq 0$ the integrand is invariant under a $[U(n)/U(n-p_1) \times U(p_1)] \times [U(n)/U(n-p_2) \times U(p_2)]$ submanifold of the coset. The integral over this manifold has to be performed exactly. Its volume is $\sim n^{p_1+p_2}$. In the replica limit of the resolvent only terms with $p = p_1 + p_2 = 0, 1$ contribute⁹.

Next, we evaluate the integral (6.5) for $N_f = 1$. We expand this determinant with respect to its first row. Using the permutation symmetry of the integration variables the square of this expansion can be reduced to only two different contributions. The partition function thus simplifies to (up to an overall constant)

$$\begin{aligned} \mathcal{Z}_{\text{QCD}_3}^{(2+2n)} = & \frac{1}{\mu^{2n}} \frac{1}{(\prod_{k=1}^{n-1} k!)^2} \int_{-1}^1 d\lambda_1 \cdots \int_{-1}^1 d\lambda_{n+1} \left[(n+1) e^{\mu(2\lambda_2 + \cdots + 2\lambda_{n+1})} \Delta^2(\lambda_2, \dots, \lambda_{n+1}) \right. \\ & \left. - n(n+1) e^{\mu(\lambda_1 + \lambda_2 + 2\lambda_3 + \cdots + 2\lambda_{n+1})} \Delta(\lambda_1, \lambda_3, \dots, \lambda_{n+1}) \Delta(\lambda_2, \lambda_3, \dots, \lambda_{n+1}) \right]. \end{aligned} \quad (6.11)$$

⁹It is important to stress that $0 \leq p_1, p_2 \leq n$ and the reader may wonder how can we make sense of sums over all saddle points in the limit $n \rightarrow 0$. What we do in this section is equivalent to a prescription given in [26, 27]. This procedure has several caveats and is not mathematically rigorous [29], but in all cases where it has been applied [26, 27, 31] it gives the correct result.

We will expand this partition function in powers of $1/\mu$. The first term in the integrand does not depend on λ_1 and therefore the integral over λ_1 of this term has to be performed exactly. It simply gives a factor 2. The remaining integrals are evolved perturbatively. We again split the integrals according to (6.7) and put $\lambda_k = \pm 1 - x_k/\mu$ depending on the upper boundary of the integral. For asymptotically large μ , the integration range of the x_k is thus given by $[0, \infty)$. In the first term the only nonvanishing possibility is $p_1 = p_2 = 0$. The term with $p_1 = p_2 = 1$ is of order n^2 and does not contribute in the replica limit. However, in the second term we have the possibilities $p_1 = 1, p_2 = 0$ and $p_1 = 0, p_2 = 1$ in addition the possibility $p_1 = p_2 = 0$. All other contributions vanish in the replica limit. We thus find that the asymptotic expansion of the QCD₃ partition function is given by

$$\begin{aligned} \mathcal{Z}_{\text{QCD}_3}^{(2+2n)} &= \frac{e^{2n\mu}}{\mu^{n^2+2n}(\prod_{k=1}^{n-1} k!)^2} \left[2(n+1) \int_0^\infty dx_2 \cdots \int_0^\infty dx_{n+1} e^{-2x_2 - \cdots - 2x_{n+1}} \Delta^2(x_2, \dots, x_{n+1}) \right. \\ &- \frac{n(n+1)}{\mu} \int_0^\infty dx_1 \cdots \int_0^\infty dx_{n+1} e^{-x_1 - x_2 - 2x_3 - \cdots - 2x_{n+1}} \Delta(x_1, x_3, \dots, x_{n+1}) \Delta(x_2, x_3, \dots, x_{n+1}) \\ &+ \frac{2n(n+1)\mu^{n-1}e^{-2\mu}}{\mu} \int_0^\infty dx_1 \cdots \int_0^\infty dx_{n+1} e^{-x_1 - x_2 - 2x_3 - \cdots - 2x_{n+1}} \\ &\quad \left. \times \Delta(2 + \frac{x_1}{\mu}, x_3, \dots, x_{n+1}) \Delta(x_2, x_3, \dots, x_{n+1}) \right]. \end{aligned} \quad (6.12)$$

We evaluate these integrals by means of the orthogonal polynomial method [44]. The orthonormal polynomials corresponding to the exponential weight function e^{-2x} are the Laguerre polynomials

$$P_k(x) = \sqrt{2} L_k(2x). \quad (6.13)$$

By adding rows and columns the Vandermonde determinants can be expressed in terms of these orthogonal polynomials. Keeping only the leading order terms in $1/\mu$ in the exponentially suppressed term, we arrive at the partition function

$$\begin{aligned} \mathcal{Z}_{\text{QCD}_3}^{(2+2n)} &= \frac{e^{2n\mu}}{2n^2 \mu^{n^2+2n}} \left[2(n+1) \int_0^\infty dx_2 \cdots \int_0^\infty dx_{n+1} e^{-2x_2 - \cdots - 2x_{n+1}} \det^2[P_k(x_2), \dots, P_k(x_{n+1})] \right. \\ &- \frac{n(n+1)}{\mu} \int_0^\infty dx_1 \cdots \int_0^\infty dx_{n+1} e^{-x_1 - x_2 - 2x_3 - \cdots - 2x_{n+1}} \\ &\quad \times \det[P_k(x_1), P_k(x_3), \dots, P_k(x_{n+1})] \det[P_k(x_2), P_k(x_3), \dots, P_k(x_{n+1})] \\ &+ \frac{2\sqrt{2}n(n+1)e^{-2\mu}(-4\mu)^{n-1}}{\mu(n-1)!} \int_0^\infty dx_1 \cdots \int_0^\infty dx_{n+1} e^{-x_1 - x_2 - 2x_3 - \cdots - 2x_{n+1}} \\ &\quad \left. \times \det[P_k(x_3), \dots, P_k(x_{n+1})] \det[P_k(x_2), P_k(x_3), \dots, P_k(x_{n+1})] \right] + \cdots, \end{aligned} \quad (6.14)$$

where the ellipses denotes higher order contribution in $1/\mu$ to the exponentially suppressed terms. Because the polynomials are normalized to one, the first integral simply gives $n!$. This can be easily seen by writing the determinants as a sum over permutations and performing the integrals by orthogonality. In the second and third integral we have to treat the integration over x_1 and x_2 separately. In the same way as for the first integral, the remaining integrals just give $(n-1)!$. To do the integrations over x_1 and x_2 we need the integral

$$\int_0^\infty dx e^{-x} P_k(x) = (-1)^k \sqrt{2}. \quad (6.15)$$

In the third integral, only the terms that contain $P_{n-1}(x_2)$ are nonvanishing. The integrals over x_1 and x_2 have to be performed separately, and, by orthogonality, the integrals over x_3, \dots, x_{n+1} give a

factor $(n-1)!$. We thus find

$$\begin{aligned}
\mathcal{Z}_{\text{QCD}_3}^{(2+2n)} &= \frac{e^{2n\mu}}{2^{n^2}\mu^{n^2+2n}} \left[2(n+1)! - \frac{(n+1)!}{\mu} \int_0^\infty dx_1 \int_0^\infty dx_2 e^{-x_1-x_2} \sum_{k=0}^{n-1} P_k(x_1) P_k(x_2) \right. \\
&\quad \left. - \frac{2\sqrt{2}(n+1)!e^{-2\mu}(4\mu)^{n-1}}{(n-1)!\mu} \int_0^\infty dx_1 \int_0^\infty dx_2 e^{-x_1-x_2} P_{n-1}(x_2) \right] \\
&= \frac{e^{2n\mu}}{2^{n^2}\mu^{n^2+2n}} 2(n+1)! \left[1 - \frac{n}{\mu} + \frac{2^{2n-1}(-1)^n \mu^{n-2} e^{-2\mu}}{(n-1)!} \right] + \dots
\end{aligned} \tag{6.16}$$

The valence quark mass dependence is thus given by

$$\begin{aligned}
\frac{\Sigma(\mu)}{\Sigma} &= \lim_{n \rightarrow 0} \frac{1}{2n} \partial_\mu \log \mathcal{Z}_{\text{QCD}_3}^{(2+2n)} \\
&= 1 - \frac{1}{\mu} + \frac{1}{2\mu^2} - \frac{e^{-2\mu}}{2\mu^2},
\end{aligned} \tag{6.17}$$

where we have calculated the replica limit of the exponentially suppressed term by writing $1/(n-1)! = n/n!$. Notice that only the leading order term in the expansion in $1/\mu$ has been included in the exponentially suppressed term.

The exact result for the microscopic spectral density for one massless flavor is given by

$$\rho_s(u) = \frac{1}{\pi} \left[1 - \frac{\sin^2 u}{u^2} \right]. \tag{6.18}$$

The corresponding valence quark mass dependence is equal to

$$\begin{aligned}
\frac{\Sigma(\mu)}{\Sigma} &= \int_0^\infty \frac{2\mu\rho_s(u)}{u^2 + \mu^2} du \\
&= 1 - \frac{1}{\mu} + \frac{1}{2\mu^2} (1 - e^{-2\mu}),
\end{aligned} \tag{6.19}$$

in perfect agreement with the replica result. All higher order contributions to the exponentially suppressed term cancel!

Let us now consider the case of two massless flavors. We again expand the determinant in (6.6) and collect terms that result in the same value for the integral as we did in the case of one massless flavor. We find

$$\begin{aligned}
\mathcal{Z}_{\text{QCD}_3}^{(4+2n)} &= \frac{1}{\mu^{4n}} \frac{1}{(\prod_{k=1}^{n-1} k!)^2} \int_{-1}^1 d\lambda_1 \dots \int_{-1}^1 d\lambda_{n+2} \left[\binom{n+2}{2} e^{\mu(2\lambda_3 + \dots + 2\lambda_{n+2})} (\lambda_2 - \lambda_1)^2 \Delta^2(\lambda_3, \dots, \lambda_{n+2}) \right. \\
&\quad - 2 \binom{n+2}{3} \binom{3}{2} (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) e^{\mu(\lambda_2 + \lambda_3 + 2\lambda_4 + \dots + 2\lambda_{n+2})} \\
&\quad \quad \times \Delta(\lambda_2, \lambda_4, \dots, \lambda_{n+2}) \Delta(\lambda_3, \lambda_4, \dots, \lambda_{n+2}) \\
&\quad + \binom{n+2}{4} \binom{4}{2} (\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3) e^{\mu(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2\lambda_5 + \dots + 2\lambda_{n+2})} \\
&\quad \quad \times \Delta(\lambda_1, \lambda_2, \lambda_5, \dots, \lambda_{n+2}) \Delta(\lambda_3, \lambda_4, \lambda_5, \dots, \lambda_{n+2}) \left. \right].
\end{aligned} \tag{6.20}$$

The integrations that are not accompanied by an exponential should be performed exactly, whereas the remaining integration variables are rescaled as $\lambda_k = \pm 1 - x_k/\mu$ with sign determined by the upper

limit of the integration in (6.7). They are performed by expanding the determinants as sums over permutations. The integrations that are accompanied by the exponential e^{2x_k} are easily evaluated by orthogonality, and give as result $n!$, $(n-1)!$ and $(n-2)!$, respectively, for the three different types of terms in (6.20). Summarizing, by keeping only the leading and first subleading $1/\mu$ corrections in the exponentially suppressed term, we find

$$\begin{aligned}
\mathcal{Z}_{\text{QCD}_3}^{(4+2n)} = & \frac{e^{2n\mu}}{\mu^{4n+n^2}2^{n^2}} \left[\binom{n+2}{2} n! \int_{-1}^1 (\lambda_1 - \lambda_2)^2 d\lambda_1 d\lambda_2 \right. \\
& - 2 \binom{n+2}{3} \binom{3}{2} \frac{(n-1)!}{\mu} \int_{-1}^1 d\lambda_1 \int_0^\infty \int_0^\infty dx_2 dx_3 (1 - \lambda_1 - \frac{x_2}{\mu})(1 - \lambda_1 - \frac{x_3}{\mu}) \\
& \quad \times \sum_{k=0}^{n-1} P_k(x_2) P_k(x_3) e^{-x_2-x_3} \\
& + \binom{n+2}{4} \binom{4}{2} (n-2)! \frac{1}{\mu^4} \int_0^\infty \cdots \int_0^\infty dx_1 \cdots dx_4 (x_2 - x_1)(x_4 - x_3) \\
& \quad \times \sum_{k < l} P_k(x_1) P_l(x_2) [P_k(x_3) P_l(x_4) - P_l(x_3) P_k(x_4)] e^{-x_1-x_2-x_3-x_4} \\
& + 4 \binom{n+2}{3} \binom{3}{2} \mu^{n-2} e^{-2\mu} (-1)^{n-1} 2^{2n-1} \int_{-1}^1 d\lambda_1 \int_0^\infty \int_0^\infty dx_2 dx_3 \\
& \quad \times (-1 - \lambda_1 - \frac{x_2}{\mu})(1 - \lambda_1 - \frac{x_3}{\mu}) e^{-x_2-x_3} \left(\frac{P_{n-1}(x_3)}{\sqrt{2}} \right. \\
& \quad \left. + \frac{n-1}{2} \frac{x_2}{\mu} \frac{P_{n-1}(x_3)}{\sqrt{2}} - \sum_{k=0}^{n-2} \frac{2k+1}{4\mu} \frac{P_{n-1}(x_3)}{\sqrt{2}} - \frac{n-1}{4\mu} \frac{P_{n-2}(x_3)}{\sqrt{2}} \right) \Big] + \dots.
\end{aligned} \tag{6.21}$$

In the last terms we have included a factor 2 to account for expanding λ_2 and λ_3 around $+1$ and -1 and vice versa. The last three terms represent the $1/\mu$ corrections to the Vandermonde determinant. They have been simplified by means of recursion relations for the Laguerre polynomials. The ellipses denote higher order corrections to the exponentially suppressed terms. To calculate the integrals we need the following result:

$$\int_0^\infty dx e^{-x} x P_k(x) = (-1)^k (2k+1) \sqrt{2}. \tag{6.22}$$

The partition function is thus given by

$$\begin{aligned}
\mathcal{Z}_{\text{QCD}_3}^{(4+2n)} = & \frac{e^{2n\mu}}{\mu^{4n+n^2}2^{n^2}} \left[\binom{n+2}{2} \frac{8n!}{3} \right. \\
& - 2 \binom{n+2}{3} \binom{3}{2} (n-1)! \left[\frac{16n}{3\mu} + \frac{8}{\mu^2} \sum_{k=0}^{n-1} (2k+1) + \frac{4}{\mu^3} \sum_{k=0}^{n-1} (2k+1)^2 \right] \\
& + \binom{n+2}{4} \binom{4}{2} \frac{8(n-2)!}{\mu^4} \sum_{k < l} ((-1)^k 2k - (-1)^l 2l)^2 \\
& - e^{-2\mu} \binom{n+2}{3} \binom{3}{2} \frac{8}{3\mu^{2-n}} \left(1 - \frac{(n-1)(2+n/4)}{\mu} \right) \Big] + \dots.
\end{aligned} \tag{6.23}$$

The sums over k and l are elementary,

$$\sum_{k=0}^{n-1} (2k+1) = n^2, \quad (6.24)$$

$$\sum_{k=0}^{n-1} k^2 = \frac{n}{6}(n-1)(2n-1), \quad (6.25)$$

$$\sum_{k=0}^{n-1} (2k+1)(-1)^k = n(-1)^{n+1}, \quad (6.26)$$

$$\sum_{k=0}^{n-1} (2k+1)^2 = \frac{4}{3}n(n^2 - \frac{1}{4}), \quad (6.27)$$

$$\sum_{k,l=0}^{n-1} [(2k+1)(-1)^k - (2k+1)(-1)^l]^2 = \frac{8}{3}n^2(n^2 - 1). \quad (6.28)$$

We thus find the partition function

$$\begin{aligned} \mathcal{Z}_{\text{QCD}_3}^{(4+2n)} &= \frac{4(n+2)!e^{2n\mu}}{2n^2 3\mu^{4n+n^2}} \left[1 - \frac{4n}{\mu} + \frac{6n^2}{\mu^2} - \frac{4}{\mu^3}n(n^2 - \frac{1}{4}) + \frac{n^2}{\mu^4}(n^2 - 1) \right. \\ &\quad \left. - \frac{ne^{-2\mu}}{\mu^2} \left(1 - \frac{(n-1)(2+n/4)}{\mu} \right) \right] + \dots \end{aligned} \quad (6.29)$$

In the replica limit this reduces to

$$\mathcal{Z}_{\text{QCD}_3}^{(4+2n)} = \frac{4e^{2n\mu}}{3\mu^{4n}} \left[1 - \frac{4n}{\mu} + \frac{n}{\mu^3} - ne^{-2\mu} \left(\frac{1}{\mu^2} + \frac{2}{\mu^3} \right) \right] + \dots, \quad (6.30)$$

which leads to the following μ -dependence of the resolvent

$$\frac{\Sigma(\mu)}{\Sigma} = 1 - \frac{2}{\mu} + \frac{2}{\mu^2} - \frac{3}{2\mu^4} + e^{-2\mu} \left(\frac{1}{\mu^2} + \frac{3}{\mu^3} + \frac{3}{2\mu^4} \right) + \dots \quad (6.31)$$

We have not calculated the terms of order $e^{-2\mu}/\mu^4$ that appear in the expansion of the partition function. By comparison with the exact result given by

$$\begin{aligned} \frac{\Sigma(\mu)}{\Sigma} &= \frac{\mu}{2} \left[I_{5/2}(\mu)K_{5/2}(\mu) + I_{3/2}(\mu)K_{3/2}(\mu) + I_{5/2}(\mu)K_{1/2}(\mu) + I_{7/2}(\mu)K_{3/2}(\mu) \right] \\ &= 1 - \frac{2}{\mu} + \frac{2}{\mu^2} - \frac{3}{2\mu^4} + e^{-2\mu} \left[\frac{1}{\mu^2} + \frac{3}{\mu^3} + \frac{3}{2\mu^4} \right], \end{aligned} \quad (6.32)$$

it turns out that such terms vanish in the replica limit (see also eq. (4.52)). To complete the calculation we will have to show that these terms, as well as higher order exponentially suppressed terms, vanish in the replica limit. The termination of the asymptotic series we have observed in this calculation is actually a more general phenomenon known as Duistermaat-Heckman localization [48, 29] (see [49] for a pedagogical discussion of this topic).

7 Conclusions

In the domain where the Compton wavelength of the Goldstone modes is much larger than the size of the box, the QCD_3 finite volume partition function for $2N_f$ flavors is given by a $U(2N_f)/U(N_f) \times U(N_f)$ unitary matrix integral. The main objective of this article has been to relate the spectrum of the QCD_3 Dirac operator to this partition function, and extensions of it with more flavors. An alternative starting point would have been the low-energy limit of the graded (or supersymmetric) generating function for the Dirac spectrum. This partition function is obtained by complementing the QCD partition function with both fermionic and bosonic “ghost quarks”. It is invariant under an internal supersymmetry. In this way correlation functions of the QCD Dirac eigenvalues can be obtained in a mathematically rigorous way. However, we have here been interested in the question whether the usual low-energy effective partition function, without bosonic ghost quarks, already contains sufficient information to completely constrain the low-energy Dirac spectrum.

This question has been answered affirmatively earlier within the context of Random Matrix Theory. The spectral k -point correlation functions are given by the ratio of the usual partition function and a partition function with $2k$ additional flavors. These additional flavors can be related to the fermionic and bosonic ghost quarks of the supersymmetric generating function. *A priori* there is no reason why the spectral correlation functions of eigenvalues in Random Matrix Theory should coincide with those of the Dirac operator in the low-energy limit of QCD. Can we reach the same conclusion without relying on Random Matrix Theory?

We have studied this question in three different ways. First, we have found exact relations between the QCD_3 partition function and the QCD_4 partition function analytically continued to half-integer topological charge. We have next added n replica quarks to the two theories and taken the replica limit. This has allowed us to derive explicit relations between the microscopic spectral density of the Dirac operator in QCD_4 and the corresponding microscopic spectral density of the Dirac operator in QCD_3 . These relations suggest the existence of not yet well-understood connections between the physics of QCD_3 and QCD_4 , and are thus of independent interest. Second, the finite volume partition function of QCD_3 satisfies Virasoro constraints. They can be solved recursively, providing us with the small mass expansion of the finite volume partition function, and a series of spectral sum rules. Third, we have directly analyzed the replica limit of the Unitary Matrix integral corresponding to the finite-volume partition functions of QCD_3 . In this way we have been able to derive the first few terms of the large mass expansion. Because the series truncate, this expansion turns out to be exact.

A more general question we have addressed is the applicability of the replica method to the calculation of quenched averages. It was suggested recently that exact non-perturbative results can be obtained using an appropriate scheme of replica symmetry breaking. The general relations we have derived between the microscopic spectral densities of QCD_3 and QCD_4 , though, follow from a completely straightforward replica limit. As an example, either by using previously obtained results for the QCD_4 partition function or by the direct replica calculation of this paper we have reproduced exact results from the large mass expansion of the partition function. Possible caveats of the replica limit do not show up in our calculation. We do not know the reasons behind this observation, and it would be interesting to obtain a better understanding of the conditions for the applicability of the replica method.

Generally, it has been expected that the replica method will not give exact results for spectral correlation functions but, at best, the coefficients of an asymptotic expansion. However, there are important cases where the series expansion turns out to *truncate*. We have witnessed a dramatic

example of this phenomenon in this article. However, there are still subtleties. The finite volume QCD_4 partition function is well-defined for all integer values ν of the topological charge. What is its analytical continuation in the complex ν -plane? A naive analytical continuation fails because of the presence of a $\sin \pi \nu$ -term. The correct analytical continuation is obtained by changing the integration contour such that translational invariance is restored. We have not calculated for general N_f the order at which the large mass expansion terminates. We believe that it is important to get a better understanding of this question and its relation with Duistermaat-Heckman localization, but this is beyond the scope of this work.

The replica method also yields the correct power-series expansion for small masses. As in QCD_4 the small mass expansion has de Wit–t Hooft poles that limit the number of terms that can be obtained this way, but up to the order the replica method can be used all results agree exactly with earlier results from both Random Matrix Theory and the supersymmetric formulation. These three very different methods are thus obviously confirming each other.

In conclusion, we have shown the existence of powerful relations between the QCD_3 and the QCD_4 partition functions. The replica limit of these results provides us with exact non-perturbative relations between the spectral correlation functions of the Dirac operator in the two theories. It would be most interesting to derive similar relations from the supersymmetric formulation as well.

Acknowledgments

This work was partially supported by the US DOE grant DE-FG-88ER40388. The work of G.A. and P.H.D. was supported in part by EU TMR grant no. ERBFMRXCT97-0122. D.D. is supported by FAPESP (Brazilian Agency). Kim Splittorff and Dominique Touban are thanked for useful discussions. Two of us (P.H.D. and J.J.M.V.) acknowledge useful discussions at the Institute of Nuclear Theory of the University of Washington at Seattle in the initial stages of this work.

A Relations between QCD₃ and QCD₄ determinants

In this Appendix we will prove the two eqs. (3.34) and (3.35) from two different lemmas, which then completes the proof of Theorem III eq. (3.27).

LEMMA 1: *Let the matrices $\mathbf{D}(\{\mu_i\}, \zeta, \omega)$, $\mathbf{C}(\{\mu_i\}, \zeta, \omega)$ and $\mathbf{S}(\{\mu_i\})$ be defined as in eqs. (3.31) and (3.14), respectively. Then the following relation holds:*

$$(\omega + \zeta) \det \mathbf{D}(\{\mu_i\}, \zeta, \omega) + (\omega - \zeta) \det \mathbf{D}(\{\mu_i\}, -\zeta, \omega) = (-1)^{N_f+1} 2^{N_f+1} \det \mathbf{C}(\{\mu_i\}, \zeta, \omega) \det \mathbf{S}(\{\mu_i\}). \quad (\text{A.1})$$

PROOF: We will show that the left hand side factorizes into the right hand side by adding and subtracting rows and columns and expanding the determinants appropriately.

Due to the block structure in the definition of the matrix \mathbf{D} in eq. (3.31) we can add the upper blocks \mathbf{C} and \mathbf{S} to the lower ones and obtain after taking out factors of two,

$$\det \mathbf{D}(\{\mu_i\}, \zeta, \omega) = 2^{N_f} \det \begin{pmatrix} \mathbf{C}(\{\mu_i\}) & -\mathbf{S}(\{\mu_i\}) \\ \mathbf{C}(\{\mu_i\}) & \mathbf{0} \\ \zeta^{j-1} \cosh^{(j-1)}(\zeta) & -\zeta^{j-1} \sinh^{(j-1)}(\zeta) \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & -\omega^{j-1} \sinh^{(j-1)}(\omega) \end{pmatrix}. \quad (\text{A.2})$$

When evaluating the determinant $\mathbf{D}(\{\mu_i\}, -\zeta, \omega)$ we can use that in the row depending on ζ the functions $\zeta^{j-1} \cosh^{(j-1)}(\zeta)$ are even under $\zeta \rightarrow -\zeta$ whereas the functions $\zeta^{j-1} \sinh^{(j-1)}(\zeta)$ are odd. In order to add the two determinants in eq. (A.1) we multiply the factors $(\omega \pm \zeta)$ into the last column of the respective determinants. Next, we use that determinants differing by only one single column or row can be added. We use this to split the left hand side of eq. (A.1) in the following way, where we explicitly display the last column,

$$\begin{aligned} (\omega + \zeta) \det \mathbf{D}(\{\mu_i\}, \zeta, \omega) &+ (\omega - \zeta) \det \mathbf{D}(\{\mu_i\}, -\zeta, \omega) = \\ = 2^{N_f} &\left[\det \begin{pmatrix} \mathbf{C}(\{\mu_i\}) & -\mathbf{S}(\{\mu_i\}) & -\omega \mu_i^{N_f} \sinh^{(N_f)}(\mu_i) \\ \mathbf{C}(\{\mu_i\}) & \mathbf{0} & 0 \\ \zeta^{j-1} \cosh^{(j-1)}(\zeta) & -\zeta^{j-1} \sinh^{(j-1)}(\zeta) & -\zeta^{N_f+1} \sinh^{(N_f)}(\zeta) \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & -\omega^{j-1} \sinh^{(j-1)}(\omega) & -\omega^{N_f+1} \sinh^{(N_f)}(\omega) \end{pmatrix} \right. \\ &+ \det \begin{pmatrix} \mathbf{C}(\{\mu_i\}) & -\mathbf{S}(\{\mu_i\}) & -\zeta \mu_i^{N_f} \sinh^{(N_f)}(\mu_i) \\ \mathbf{C}(\{\mu_i\}) & \mathbf{0} & 0 \\ \zeta^{j-1} \cosh^{(j-1)}(\zeta) & -\zeta^{j-1} \sinh^{(j-1)}(\zeta) & -\omega \zeta^{N_f} \sinh^{(N_f)}(\zeta) \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & -\omega^{j-1} \sinh^{(j-1)}(\omega) & -\zeta \omega^{N_f} \sinh^{(N_f)}(\omega) \end{pmatrix} \\ &+ \det \begin{pmatrix} \mathbf{C}(\{\mu_i\}) & -\mathbf{S}(\{\mu_i\}) & -\omega \mu_i^{N_f} \sinh^{(N_f)}(\mu_i) \\ \mathbf{C}(\{\mu_i\}) & \mathbf{0} & 0 \\ \zeta^{j-1} \cosh^{(j-1)}(\zeta) & \zeta^{j-1} \sinh^{(j-1)}(\zeta) & -\zeta^{N_f+1} \sinh^{(N_f)}(\zeta) \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & -\omega^{j-1} \sinh^{(j-1)}(\omega) & -\omega^{N_f+1} \sinh^{(N_f)}(\omega) \end{pmatrix} \\ &\left. + \det \begin{pmatrix} \mathbf{C}(\{\mu_i\}) & -\mathbf{S}(\{\mu_i\}) & \zeta \mu_i^{N_f} \sinh^{(N_f)}(\mu_i) \\ \mathbf{C}(\{\mu_i\}) & \mathbf{0} & 0 \\ \zeta^{j-1} \cosh^{(j-1)}(\zeta) & \zeta^{j-1} \sinh^{(j-1)}(\zeta) & \omega \zeta^{N_f} \sinh^{(N_f)}(\zeta) \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & -\omega^{j-1} \sinh^{(j-1)}(\omega) & \zeta \omega^{N_f} \sinh^{(N_f)}(\omega) \end{pmatrix} \right]. \quad (\text{A.3}) \end{aligned}$$

In the next step we can add the first and the third determinant on the right hand side, since they only differ by the last but one row. And we can also add the second and forth determinant for the same

reason, after multiplying in the fourth determinant the last column and last but one row with (-1) . We find

$$\begin{aligned}
& (\omega + \zeta) \det \mathbf{D}(\{\mu_i\}, \zeta, \omega) + (\omega - \zeta) \det \mathbf{D}(\{\mu_i\}, -\zeta, \omega) = \\
& = 2^{N_f+1} \left[\det \begin{pmatrix} \mathbf{C}(\{\mu_i\}) & -\mathbf{S}(\{\mu_i\}) & 0 \\ \mathbf{C}(\{\mu_i\}) & \mathbf{0} & 0 \\ \zeta^{j-1} \cosh^{(j-1)}(\zeta) & 0 & -\zeta^{N_f+1} \sinh^{(N_f)}(\zeta) \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & -\omega^{j-1} \sinh^{(j-1)}(\omega) & -\omega^{N_f+1} \sinh^{(N_f)}(\omega) \end{pmatrix} \right. \\
& + \det \begin{pmatrix} \mathbf{C}(\{\mu_i\}) & -\mathbf{S}(\{\mu_i\}) & -\omega \mu_i^{N_f} \sinh^{(N_f)}(\mu_i) \\ \mathbf{C}(\{\mu_i\}) & \mathbf{0} & 0 \\ \zeta^{j-1} \cosh^{(j-1)}(\zeta) & 0 & 0 \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & -\omega^{j-1} \sinh^{(j-1)}(\omega) & 0 \end{pmatrix} \\
& \left. + \det \begin{pmatrix} \mathbf{C}(\{\mu_i\}) & -\mathbf{S}(\{\mu_i\}) & -\zeta \mu_i^{N_f} \sinh^{(N_f)}(\mu_i) \\ \mathbf{C}(\{\mu_i\}) & \mathbf{0} & 0 \\ 0 & -\zeta^{j-1} \sinh^{(j-1)}(\zeta) & 0 \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & -\omega^{j-1} \sinh^{(j-1)}(\omega) & -\zeta \omega^{N_f} \sinh^{(N_f)}(\omega) \end{pmatrix} \right], \tag{A.4}
\end{aligned}$$

where we have again split the determinant, resulting from the first addition, into two parts with respect to the last column, and taken out a factor of 2. The first determinant in the last equation is already part of the desired result. In the last column we can write $\sinh^{(N_f)} = \cosh^{(N_f+1)}$. If we then permute the last column N_f times to the left the first $N_f + 2$ columns only contain derivatives of \cosh . We Laplace-expand the resulting determinant into products of $(N_f + 2) \times (N_f + 2)$ blocks and $N_f \times N_f$ blocks. After taking out all signs we obtain

$$(-1)^{N_f+1} 2^{N_f+1} \det \begin{pmatrix} \mathbf{C}(\{\mu_i\}) & 0 \\ \zeta^{j-1} \cosh^{(j-1)}(\zeta) & \zeta^{N_f+1} \cosh^{(N_f+1)}(\zeta) \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & \omega^{N_f+1} \cosh^{(N_f+1)}(\omega) \end{pmatrix} \det \mathbf{S}(\{\mu_i\}). \tag{A.5}$$

What remains to be shown is that the two remaining determinants in eq. (A.4) give rise to the same term as in eq. (A.5), except that the last column vector is reading $(\mu_i^{N_f+1} \cosh^{(N_f+1)}(\mu_i), 0, 0)$. In that way they will add up to the right hand side of Lemma 1, eq. (A.1).

To proceed, we find that in the third determinant of eq. (A.4), the last term in the last column can be put equal to zero. To see this we expand the determinant with respect to the last column and note that the determinant multiplying $-\zeta \omega^{N_f} \sinh^{(N_f)}(\omega)$ vanishes, which can be easily seen by expanding it into products of $(N_f + 1) \times (N_f + 1)$ blocks and $N_f \times N_f$ blocks. Next, in both the second and third determinant in eq. (A.4) we take the common factors ζ and ω out of the last column, respectively, and multiply them into the last but one row. The two determinants are then equal up to precisely the last but one row and can thus be added. We arrive at

$$2^{N_f+1} \det \begin{pmatrix} \mathbf{C}(\{\mu_i\}) & -\mathbf{S}(\{\mu_i\}) & -\mu_i^{N_f} \sinh^{(N_f)}(\mu_i) \\ \mathbf{C}(\{\mu_i\}) & \mathbf{0} & 0 \\ \omega \zeta^{j-1} \cosh^{(j-1)}(\zeta) & -\zeta^j \sinh^{(j-1)}(\zeta) & 0 \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & -\omega^{j-1} \sinh^{(j-1)}(\omega) & 0 \end{pmatrix}. \tag{A.6}$$

It is straightforward to see that, when expanding with respect to the last column, and then into products of $(N_f + 1) \times (N_f + 1)$ blocks of \mathbf{C} and $N_f \times N_f$ of \mathbf{S} , the ω multiplying the left side of the

last but one row can be multiplied into the right side of the last row

$$2^{N_f+1} \det \begin{pmatrix} \mathbf{C}(\{\mu_i\}) & -\mathbf{S}(\{\mu_i\}) & -\mu_i^{N_f} \sinh^{(N_f)}(\mu_i) \\ \mathbf{C}(\{\mu_i\}) & \mathbf{0} & 0 \\ \zeta^{j-1} \cosh^{(j-1)}(\zeta) & -\zeta^j \sinh^{(j-1)}(\zeta) & 0 \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & -\omega^j \sinh^{(j-1)}(\omega) & 0 \end{pmatrix}. \quad (\text{A.7})$$

As we have said before we claim that this is equal to the following product:

$$(-1)^{N_f+1} 2^{N_f+1} \det \begin{pmatrix} \mathbf{C}(\{\mu_i\}) & \mu_i^{N_f+1} \cosh^{(N_f+1)}(\mu_i) \\ \zeta^{j-1} \cosh^{(j-1)}(\zeta) & 0 \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & 0 \end{pmatrix} \det \mathbf{S}(\{\mu_i\}), \quad (\text{A.8})$$

which will add up to eq. (A.5) in order to give the desired right hand side of eq. (A.1). To see this we still have to do two more Laplace expansions. First of all we eliminate in eq. (A.7) the upper block \mathbf{C} by subtracting the lower one. Then, we permute the first column N_f times to the right to obtain

$$-2^{N_f+1} \det \begin{pmatrix} \mathbf{0} & 0 & \mathbf{S}(\{\mu_i\}) & \mu_i^{N_f} \sinh^{(N_f)}(\mu_i) \\ \mu_i^j \cosh^{(j)}(\mu_i) & \cosh(\mu_i) & \mathbf{0} & 0 \\ \zeta^j \cosh^{(j)}(\zeta) & \cosh(\zeta) & \zeta^j \sinh^{(j-1)}(\zeta) & 0 \\ \omega^j \cosh^{(j)}(\omega) & \cosh(\omega) & \omega^j \sinh^{(j-1)}(\omega) & 0 \end{pmatrix}, \quad (\text{A.9})$$

after taking out all signs. Our strategy will now be as follows. We will expand this determinant into products of $N_f \times N_f$ blocks and $(N_f + 2) \times (N_f + 2)$ blocks. The first blocks, after taking out common factors and rewriting $\cosh^{(j)} = \sinh^{(j-1)}$ will give rise to the desired prefactor $\det \mathbf{S}(\{\mu_i\})$. In doing so we obtain for eq. (A.9)

$$\begin{aligned} & (-1)^{N_f+1} 2^{N_f+1} \left\{ \det \left(\mu_i^j \cosh^{(j)}(\mu_i) \right) \det \begin{pmatrix} 0 & \mathbf{S}(\{\mu_i\}) & \mu_i^{N_f} \sinh^{(N_f)}(\mu_i) \\ \cosh(\zeta) & \zeta^j \sinh^{(j-1)}(\zeta) & 0 \\ \cosh(\omega) & \omega^j \sinh^{(j-1)}(\omega) & 0 \end{pmatrix} \right. \\ & - \sum_{k=1}^{N_f} (-1)^k \left[\det \begin{pmatrix} \mu_i^j \cosh^{(j)}(\mu_i) \\ \dots \text{no } k \dots \\ \omega^j \cosh^{(j)}(\omega) \end{pmatrix} \det \begin{pmatrix} \cosh(\mu_k) & \mathbf{0} & 0 \\ 0 & \mathbf{S}(\{\mu_i\}) & \mu_i^{N_f} \sinh^{(N_f)}(\mu_i) \\ \cosh(\zeta) & \zeta^j \sinh^{(j-1)}(\zeta) & 0 \end{pmatrix} \right. \\ & \left. \left. - \det \begin{pmatrix} \mu_i^j \cosh^{(j)}(\mu_i) \\ \dots \text{no } k \dots \\ \zeta^j \cosh^{(j)}(\zeta) \end{pmatrix} \det \begin{pmatrix} \cosh(\mu_k) & \mathbf{0} & 0 \\ 0 & \mathbf{S}(\{\mu_i\}) & \mu_i^{N_f} \sinh^{(N_f)}(\mu_i) \\ \cosh(\omega) & \omega^j \sinh^{(j-1)}(\omega) & 0 \end{pmatrix} \right] \right\}. \quad (\text{A.10}) \end{aligned}$$

The first product gives already part of eq. (A.8) whereas the remaining contributions originate from the other terms. To see that we take out $\prod_i \mu_i$ out of the first determinant and multiply the i th row of the second determinant with μ_i (for $i = 1, \dots, N_f$). Then, we write $\cosh^{(j)} = \sinh^{(j-1)}$ in the first and $\sinh^{(j-1)} = \cosh^{(j)}$ in the second factor to obtain

$$(-1)^{N_f+1} 2^{N_f+1} \det \mathbf{S}(\{\mu_i\}) \det \begin{pmatrix} 0 & \mu_i^j \cosh^{(j)}(\mu_i) & \mu_i^{N_f+1} \cosh^{(N_f+1)}(\mu_i) \\ \cosh(\zeta) & \zeta^j \cosh^{(j)}(\zeta) & 0 \\ \cosh(\omega) & \omega^j \cosh^{(j)}(\omega) & 0 \end{pmatrix}. \quad (\text{A.11})$$

We are still left with the sum over k in eq. (A.10). To simplify it further we take out common factors $\mu_{i \neq k}$ from the first determinant as well as rewrite $\cosh^{(j)} = \sinh^{(j-1)}$. We then expand the second

determinant into products of 2×2 blocks and $N_f \times N_f$ blocks, where the 2×2 blocks consist of the first and last column

$$\begin{aligned}
& 2^{N_f+1} \sum_{k,l=1}^{N_f} (-1)^{k+l-1} \left[\det \begin{pmatrix} \mu_i^j \sinh^{(j-1)}(\mu_i) \\ \dots \text{no } k \dots \\ \omega^j \sinh^{(j-1)}(\omega) \end{pmatrix} \cosh(\mu_k) \mu_l^{N_f} \sinh^{(N_f)}(\mu_l) \det \begin{pmatrix} \mathbf{S}(\{\mu_{i \neq l}\}) \\ \zeta^j \sinh^{(j-1)}(\zeta) \end{pmatrix} \right. \\
& \quad \left. - \det \begin{pmatrix} \mu_i^j \sinh^{(j-1)}(\mu_i) \\ \dots \text{no } k \dots \\ \zeta^j \sinh^{(j-1)}(\zeta) \end{pmatrix} \cosh(\mu_k) \mu_l^{N_f} \sinh^{(N_f)}(\mu_l) \det \begin{pmatrix} \mathbf{S}(\{\mu_{i \neq l}\}) \\ \omega^j \sinh^{(j-1)}(\omega) \end{pmatrix} \right] \\
&= 2^{N_f+1} \left[\sum_{k,l=1}^{N_f} (-1)^{k+l-1} \zeta \omega \prod_{j \neq k,l} \mu_j \cosh(\mu_k) \mu_l^{N_f+1} \cosh^{(N_f+1)}(\mu_l) \right. \\
& \quad \times (\det \mathbf{S}(\{\mu_{i \neq k}\}, \omega) \det \mathbf{S}(\{\mu_{i \neq l}\}, \zeta) - \det \mathbf{S}(\{\mu_{i \neq k}\}, \zeta) \det \mathbf{S}(\{\mu_{i \neq l}\}, \omega)) \Big] \\
&= 2^{N_f+1} \left[\sum_{k>l}^{N_f} (-1)^{k+l-1} \zeta \omega \prod_{j \neq k,l} \mu_j \cosh(\mu_k) \mu_l^{N_f+1} \cosh^{(N_f+1)}(\mu_l) \det \mathbf{S}(\{\mu_i\}) \det \mathbf{S}(\{\mu_{i \neq l,k}\}, \zeta, \omega) \right. \\
& \quad \left. - \sum_{k<l}^{N_f} (-1)^{k+l-1} \zeta \omega \prod_{j \neq k,l} \mu_j \cosh(\mu_k) \mu_l^{N_f+1} \cosh^{(N_f+1)}(\mu_l) \det \mathbf{S}(\{\mu_i\}) \det \mathbf{S}(\{\mu_{i \neq k,l}\}, \zeta, \omega) \right] \\
&= 2^{N_f+1} \det \mathbf{S}(\{\mu_i\}) \\
& \quad \times \sum_{k>l}^{N_f} (-1)^{k+l-1} \zeta \omega \prod_{j \neq k,l} \mu_j \det \begin{pmatrix} \cosh(\mu_k) & \mu_k^{N_f+1} \cosh^{(N_f+1)}(\mu_k) \\ \cosh(\mu_l) & \mu_l^{N_f+1} \cosh^{(N_f+1)}(\mu_l) \end{pmatrix} \det \mathbf{S}(\{\mu_{i \neq k,l}\}, \zeta, \omega) \\
&= (-1)^{N_f+1} 2^{N_f+1} \det \mathbf{S}(\{\mu_i\}) \det \begin{pmatrix} \cosh(\mu_i) & \mu_i^j \cosh^{(j)}(\mu_i) & \mu_i^{N_f+1} \cosh^{(N_f+1)}(\mu_i) \\ 0 & \zeta^j \cosh^{(j)}(\zeta) & 0 \\ 0 & \omega^j \cosh^{(j)}(\omega) & 0 \end{pmatrix}. \quad (\text{A.12})
\end{aligned}$$

In the first step we have only taken out common factors ζ and ω in order to be able to use the short notation \mathbf{S} . In the second step we have used a property for antisymmetric products of determinants. A proof of the relations used can be found for example in Lemma 5 of [35]. Moreover, we have to distinguish the cases $k > l$ and $k < l$, whereas the terms with $k = l$ drop out. In step three we have rewritten everything in terms of determinants. This is just to see that the double sum gives the Laplace expansion of the matrix in the last line. Here, we have again used that $\sinh^{(j-1)} = \cosh^{(j)}$ and multiplied the factor $\zeta \omega \prod_{j \neq k,l} \mu_j$ into the last determinant $\det \mathbf{S}(\{\mu_{i \neq k,l}\}, \zeta, \omega)$. To summarize our efforts, the last line of eq. (A.12) together with eq. (A.11) sums up to eq. (A.8), as we wished to show. So we see that starting with the left hand side of eq. (A.3) we arrive at eqs. (A.5) and (A.8), which add up to the right hand side of the lemma eq. (A.1).

LEMMA 2: *With the same definitions as in Lemma 1 the following relation holds:*

$$(\omega + \zeta) \det \mathbf{D}(\{\mu_i\}, \zeta, \omega) - (\omega - \zeta) \det \mathbf{D}(\{\mu_i\}, -\zeta, \omega) = (-1)^{N_f} 2^{N_f+1} \det \mathbf{S}(\{\mu_i\}, \zeta, \omega) \det \mathbf{C}(\{\mu_i\}). \quad (\text{A.13})$$

PROOF: The proof will go very much along the same lines as in Lemma 1. However, we will arrive at a determinant of \mathbf{S} of size $(N_f + 2)$ instead of size N_f , which requires a different addition and subtraction scheme of rows and columns than in the proof of Lemma 1.

We start again from the definition of the matrix \mathbf{D} in eq. (3.31) and this time subtract the upper blocks \mathbf{C} and \mathbf{S} from the lower ones,

$$\det \mathbf{D}(\{\mu_i\}, \zeta, \omega) = 2^{N_f} \det \begin{pmatrix} \mathbf{0} & -\mathbf{S}(\{\mu_i\}) \\ \mathbf{C}(\{\mu_i\}) & \mathbf{S}(\{\mu_i\}) \\ \zeta^{j-1} \cosh^{(j-1)}(\zeta) & -\zeta^{j-1} \sinh^{(j-1)}(\zeta) \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & -\omega^{j-1} \sinh^{(j-1)}(\omega) \end{pmatrix}. \quad (\text{A.14})$$

If we multiply the factors $(\omega \pm \zeta)$ into the $(N_f + 1)$ -st column of the determinants and split them up the same way as before we obtain

$$\begin{aligned} (\omega + \zeta) \det \mathbf{D}(\{\mu_i\}, \zeta, \omega) & - (\omega - \zeta) \det \mathbf{D}(\{\mu_i\}, -\zeta, \omega) = \\ & = 2^{N_f} \left[\det \begin{pmatrix} \mathbf{0} & 0 & -\mathbf{S}(\{\mu_i\}) \\ \mathbf{C}(\{\mu_i\}) & \omega \mu_i^{N_f} \cosh^{(N_f)}(\mu_i) & \mathbf{S}(\{\mu_i\}) \\ \zeta^{j-1} \cosh^{(j-1)}(\zeta) & \zeta^{N_f+1} \cosh^{(N_f)}(\zeta) & -\zeta^{j-1} \sinh^{(j-1)}(\zeta) \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & \omega^{N_f+1} \cosh^{(N_f)}(\omega) & -\omega^{j-1} \sinh^{(j-1)}(\omega) \end{pmatrix} \right. \\ & + \det \begin{pmatrix} \mathbf{0} & 0 & -\mathbf{S}(\{\mu_i\}) \\ \mathbf{C}(\{\mu_i\}) & \zeta \mu_i^{N_f} \cosh^{(N_f)}(\mu_i) & \mathbf{S}(\{\mu_i\}) \\ \zeta^{j-1} \cosh^{(j-1)}(\zeta) & \omega \zeta^{N_f} \cosh^{(N_f)}(\zeta) & -\zeta^{j-1} \sinh^{(j-1)}(\zeta) \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & \zeta \omega^{N_f} \cosh^{(N_f)}(\omega) & -\omega^{j-1} \sinh^{(j-1)}(\omega) \end{pmatrix} \\ & - \det \begin{pmatrix} \mathbf{0} & 0 & -\mathbf{S}(\{\mu_i\}) \\ \mathbf{C}(\{\mu_i\}) & \omega \mu_i^{N_f} \cosh^{(N_f)}(\mu_i) & \mathbf{S}(\{\mu_i\}) \\ \zeta^{j-1} \cosh^{(j-1)}(\zeta) & -\zeta^{N_f+1} \cosh^{(N_f)}(\zeta) & \zeta^{j-1} \sinh^{(j-1)}(\zeta) \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & \omega^{N_f+1} \cosh^{(N_f)}(\omega) & -\omega^{j-1} \sinh^{(j-1)}(\omega) \end{pmatrix} \\ & \left. - \det \begin{pmatrix} \mathbf{0} & 0 & -\mathbf{S}(\{\mu_i\}) \\ \mathbf{C}(\{\mu_i\}) & -\zeta \mu_i^{N_f} \cosh^{(N_f)}(\mu_i) & \mathbf{S}(\{\mu_i\}) \\ \zeta^{j-1} \cosh^{(j-1)}(\zeta) & \omega \zeta^{N_f} \cosh^{(N_f)}(\zeta) & \zeta^{j-1} \sinh^{(j-1)}(\zeta) \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & -\zeta \omega^{N_f} \cosh^{(N_f)}(\omega) & -\omega^{j-1} \sinh^{(j-1)}(\omega) \end{pmatrix} \right]. \quad (\text{A.15}) \end{aligned}$$

In the next step we can again add the first and third determinant as well as the second and fourth determinant, after multiplying in the fourth determinant the last column and last but one row with (-1) . We obtain

$$\begin{aligned} (\omega + \zeta) \det \mathbf{D}(\{\mu_i\}, \zeta, \omega) & - (\omega - \zeta) \det \mathbf{D}(\{\mu_i\}, -\zeta, \omega) = \\ & = 2^{N_f+1} \left[\det \begin{pmatrix} \mathbf{0} & 0 & -\mathbf{S}(\{\mu_i\}) \\ \mathbf{C}(\{\mu_i\}) & 0 & \mathbf{S}(\{\mu_i\}) \\ 0 & \zeta^{N_f+1} \cosh^{(N_f)}(\zeta) & -\zeta^{j-1} \sinh^{(j-1)}(\zeta) \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & \omega^{N_f+1} \cosh^{(N_f)}(\omega) & -\omega^{j-1} \sinh^{(j-1)}(\omega) \end{pmatrix} \right. \\ & + \det \begin{pmatrix} \mathbf{0} & 0 & -\mathbf{S}(\{\mu_i\}) \\ \mathbf{C}(\{\mu_i\}) & \omega \mu_i^{N_f} \cosh^{(N_f)}(\mu_i) & \mathbf{S}(\{\mu_i\}) \\ 0 & 0 & -\zeta^{j-1} \sinh^{(j-1)}(\zeta) \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & 0 & -\omega^{j-1} \sinh^{(j-1)}(\omega) \end{pmatrix} \\ & \left. + \det \begin{pmatrix} \mathbf{0} & 0 & -\mathbf{S}(\{\mu_i\}) \\ \mathbf{C}(\{\mu_i\}) & \zeta \mu_i^{N_f} \cosh^{(N_f)}(\mu_i) & \mathbf{S}(\{\mu_i\}) \\ \zeta^{j-1} \cosh^{(j-1)}(\zeta) & 0 & 0 \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & \zeta \omega^{N_f} \cosh^{(N_f)}(\omega) & -\omega^{j-1} \sinh^{(j-1)}(\omega) \end{pmatrix} \right], \quad (\text{A.16}) \end{aligned}$$

where we have split the determinant resulting out of the first addition into two, with respect to the last column, and taken out a factor of 2. The first determinant in the last equation is already part of the desired result. In the $(N_f + 1)$ -st column we can write $\cosh^{(N_f)} = \sinh^{(N_f+1)}$. If we then permute it to the last column, the last $N_f + 2$ columns only contain derivatives of \sinh . We then Laplace-expand the resulting determinant into products of $N_f \times N_f$ blocks and $(N_f + 2) \times (N_f + 2)$ blocks and obtain

$$(-)^{N_f} 2^{N_f+1} \det \mathbf{C}(\{\mu_i\}) \det \begin{pmatrix} \mathbf{S}(\{\mu_i\}) & 0 \\ \zeta^{j-1} \sinh^{(j-1)}(\zeta) & \zeta^{N_f+1} \sinh^{(N_f+1)}(\zeta) \\ \omega^{j-1} \sinh^{(j-1)}(\omega) & \omega^{N_f+1} \sinh^{(N_f+1)}(\omega) \end{pmatrix} \quad (\text{A.17})$$

after re-arranging all signs. What remains to be shown is that the two remaining determinants in eq. (A.16) give rise to the same determinant as in eq. (A.17), with the last column vector reading $(\mu_i^{N_f+1} \sinh^{(N_f+1)}(\mu_i), 0, 0)$ instead.

We find again that in the third determinant in eq. (A.16), the last term in the $(N_f + 1)$ -st column can be set to zero following the same argument as before. Next, in both the second and third determinant in eq. (A.16) we take the common factors ζ and ω out of the $(N_f + 1)$ -st column, respectively, and multiply them into the last but one row. The two determinants are then equal up to precisely the last but one row and can thus be added,

$$2^{N_f+1} \det \begin{pmatrix} \mathbf{0} & 0 & -\mathbf{S}(\{\mu_i\}) \\ \mathbf{C}(\{\mu_i\}) & \mu_i^{N_f} \cosh^{(N_f)}(\mu_i) & \mathbf{S}(\{\mu_i\}) \\ \zeta^j \cosh^{(j-1)}(\zeta) & 0 & -\omega \zeta^j \sinh^{(j-1)}(\zeta) \\ \omega^{j-1} \cosh^{(j-1)}(\omega) & 0 & -\omega^{j-1} \sinh^{(j-1)}(\omega) \end{pmatrix}. \quad (\text{A.18})$$

Along the lines of Lemma 1 it is again straightforward to see, that the factor ω multiplying the right side of the last but one row can be multiplied into the left side of the last row

$$2^{N_f+1} \det \begin{pmatrix} \mathbf{0} & 0 & -\mathbf{S}(\{\mu_i\}) \\ \mathbf{C}(\{\mu_i\}) & \mu_i^{N_f} \cosh^{(N_f)}(\mu_i) & \mathbf{S}(\{\mu_i\}) \\ \zeta^j \cosh^{(j-1)}(\zeta) & 0 & -\zeta^{j-1} \sinh^{(j-1)}(\zeta) \\ \omega^j \cosh^{(j-1)}(\omega) & 0 & -\omega^{j-1} \sinh^{(j-1)}(\omega) \end{pmatrix}. \quad (\text{A.19})$$

This remains to be shown to be equal to the following product:

$$(-)^{N_f} 2^{N_f+1} \det \mathbf{C}(\{\mu_i\}) \det \begin{pmatrix} \mathbf{S}(\{\mu_i\}) & \mu_i^{N_f+1} \sinh^{(N_f+1)}(\mu_i) \\ \zeta^{j-1} \sinh^{(j-1)}(\zeta) & 0 \\ \omega^{j-1} \sinh^{(j-1)}(\omega) & 0 \end{pmatrix}, \quad (\text{A.20})$$

which then adds up to eq. (A.17) to give eq. (A.13). To proceed, we eliminate in eq. (A.19) the lower block \mathbf{S} by adding the upper one. Next, we expand the determinant into products of $(N_f + 2) \times (N_f + 2)$ blocks and $N_f \times N_f$ blocks. This time it will be the last blocks, that give us the prefactor $\det \mathbf{C}(\{\mu_i\})$

$$\begin{aligned} & -2^{N_f+1} \left\{ \det \begin{pmatrix} \mathbf{C}(\{\mu_i\}) & \mu_i^{N_f} \cosh^{(N_f)}(\mu_i) & 0 \\ \zeta^j \cosh^{(j-1)}(\zeta) & 0 & \sinh(\zeta) \\ \omega^j \cosh^{(j-1)}(\omega) & 0 & \sinh(\omega) \end{pmatrix} \det \left(\mu_i^j \sinh^{(j)}(\mu_i) \right) \right. \\ & - \sum_{k=1}^{N_f} (-1)^k \left[\det \begin{pmatrix} 0 & 0 & \sinh(\mu_k) \\ \mathbf{C}(\{\mu_i\}) & \mu_i^{N_f} \cosh^{(N_f)}(\mu_i) & 0 \\ \zeta^j \sinh^{(j-1)}(\zeta) & 0 & \sinh(\zeta) \end{pmatrix} \det \begin{pmatrix} \mu_i^j \sinh^{(j)}(\mu_i) \\ \dots \text{no } k \dots \\ \omega^j \sinh^{(j)}(\omega) \end{pmatrix} \right. \\ & \left. \left. - \det \begin{pmatrix} 0 & 0 & \sinh(\mu_k) \\ \mathbf{C}(\{\mu_i\}) & \mu_i^{N_f} \cosh^{(N_f)}(\mu_i) & 0 \\ \omega^j \sinh^{(j-1)}(\omega) & 0 & \sinh(\omega) \end{pmatrix} \det \begin{pmatrix} \mu_i^j \sinh^{(j)}(\mu_i) \\ \dots \text{no } k \dots \\ \zeta^j \sinh^{(j)}(\zeta) \end{pmatrix} \right] \right\}. \quad (\text{A.21}) \end{aligned}$$

The first product gives part of eq. (A.20) by taking $\prod_i \mu_i$ out of the second determinant and multiplying the first N_f rows of the first determinant, upon rewriting $\cosh^{(j)} = \sinh^{(j-1)}$,

$$(-)^{N_f} 2^{N_f+1} \det \begin{pmatrix} 0 & \mu_i^j \sinh^{(j)}(\mu_i) & \mu_i^{N_f+1} \sinh^{(N_f+1)}(\mu_i) \\ \sinh(\zeta) & \zeta^j \sinh^{(j)}(\zeta) & 0 \\ \sinh(\omega) & \omega^j \sinh^{(j)}(\omega) & 0 \end{pmatrix} \det \mathbf{C}(\{\mu_i\}) . \quad (\text{A.22})$$

Next, we tackle the sum over k in eq. (A.21). Along the lines of Lemma 1 we expand the first determinant into products of $N_f \times N_f$ blocks and 2×2 blocks,

$$\begin{aligned} & 2^{N_f+1} \sum_{k,l=1}^{N_f} (-1)^{k+l} \left[\det \begin{pmatrix} \mathbf{C}(\{\mu_{i \neq l}\}) \\ \zeta^j \cosh^{(j-1)}(\zeta) \end{pmatrix} \sinh(\mu_k) \mu_l^{N_f} \cosh^{(N_f)}(\mu_l) \det \begin{pmatrix} \mu_i^j \cosh^{(j-1)}(\mu_i) \\ \dots \text{no } k \dots \\ \omega^j \cosh^{(j-1)}(\omega) \end{pmatrix} \right. \\ & \quad \left. - \det \begin{pmatrix} \mathbf{C}(\{\mu_{i \neq l}\}) \\ \omega^j \cosh^{(j-1)}(\omega) \end{pmatrix} \sinh(\mu_k) \mu_l^{N_f} \cosh^{(N_f)}(\mu_l) \det \begin{pmatrix} \mu_i^j \cosh^{(j-1)}(\mu_i) \\ \dots \text{no } k \dots \\ \zeta^j \cosh^{(j-1)}(\zeta) \end{pmatrix} \right] \\ &= 2^{N_f+1} \left[\sum_{k,l=1}^{N_f} (-1)^{k+l} \zeta \omega \prod_{j \neq k,l} \mu_j \sinh(\mu_k) \mu_l^{N_f+1} \sinh^{(N_f+1)}(\mu_l) \right. \\ & \quad \left. \times (\det \mathbf{C}(\{\mu_{i \neq k}\}, \omega) \det \mathbf{C}(\{\mu_{i \neq l}\}, \zeta) - \det \mathbf{C}(\{\mu_{i \neq k}\}, \zeta) \det \mathbf{C}(\{\mu_{i \neq l}\}, \omega)) \right] \\ &= 2^{N_f+1} \left[\sum_{k>l}^{N_f} (-1)^{k+l} \zeta \omega \prod_{j \neq k,l} \mu_j \sinh(\mu_k) \mu_l^{N_f+1} \sinh^{(N_f+1)}(\mu_l) \det \mathbf{C}(\{\mu_i\}) \det \mathbf{C}(\{\mu_{i \neq k,l}\}, \zeta, \omega) \right. \\ & \quad \left. - \sum_{k<l}^{N_f} (-1)^{k+l} \zeta \omega \prod_{j \neq k,l} \mu_j \sinh(\mu_k) \mu_l^{N_f+1} \sinh^{(N_f+1)}(\mu_l) \det \mathbf{C}(\{\mu_i\}) \det \mathbf{C}(\{\mu_{i \neq k,l}\}, \zeta, \omega) \right] \\ &= 2^{N_f+1} \det \mathbf{C}(\{\mu_i\}) \\ & \quad \times \sum_{k>l}^{N_f} (-1)^{k+l} \zeta \omega \prod_{j \neq k,l} \mu_j \det \begin{pmatrix} \sinh(\mu_k) & \mu_k^{N_f+1} \sinh^{(N_f+1)}(\mu_k) \\ \sinh(\mu_l) & \mu_l^{N_f+1} \sinh^{(N_f+1)}(\mu_l) \end{pmatrix} \det \mathbf{C}(\{\mu_{i \neq k,l}\}, \zeta, \omega) \\ &= (-1)^{N_f} 2^{N_f+1} \det \mathbf{C}(\{\mu_i\}) \det \begin{pmatrix} \sinh(\mu_i) & \mu_i^j \sinh^{(j)}(\mu_i) & \mu_i^{N_f+1} \sinh^{(N_f+1)}(\mu_i) \\ 0 & \zeta^j \sinh^{(j)}(\zeta) & 0 \\ 0 & \omega^j \sinh^{(j)}(\omega) & 0 \end{pmatrix} . \quad (\text{A.23}) \end{aligned}$$

Here, we have performed the same steps as in the respective part of the proof of Lemma 1. Thus eqs. (A.23) and (A.22) sum up to eq. (A.20), which together with eq. (A.17) leads to the right hand side of eq. (A.13).

B Relations between QCD₃ and QCD₄ from Random Matrix Theory

In this subsection we show that the relations of Theorem III also can be derived from an entirely different point of view, based on the Random Matrix Theory representations of the QCD₃ and QCD₄ finite-volume partition functions.

Using the results of refs. [2, 4], we write the partition functions (2.7) and (2.12) as the large- \mathcal{N}

limits of the matrix integrals

$$\tilde{\mathcal{Z}}_{\text{QCD}_3}^{(2N_f)}(\{m_i\}) = \int d\phi \prod_{f=1}^{2N_f} \det(\phi + im_f) \exp \left[-\mathcal{N} \text{tr} V(\phi^2) \right] , \quad (\text{B.1})$$

for QCD₃, and

$$\tilde{\mathcal{Z}}_\nu^{(N_f)}(\{m_i\}) = \int dW \prod_{f=1}^{N_f} \det(i\bar{\phi} + m_f) \exp \left[-\frac{\mathcal{N}}{2} \text{tr} V(\bar{\phi}^2) \right] , \quad (\text{B.2})$$

for QCD₄. Here ϕ is a Hermitian $2\mathcal{N} \times 2\mathcal{N}$ matrix and $\bar{\phi}$

$$\bar{\phi} = \begin{pmatrix} 0 & W^\dagger \\ W & 0 \end{pmatrix} , \quad (\text{B.3})$$

with W an arbitrary complex matrix of size $(\mathcal{N} + \nu) \times \mathcal{N}$. In both cases the Random Matrix Theory potentials $V(\phi^2)$ are not constrained beyond giving convergent integrals with a non-vanishing spectral density at the origin, $\rho(0) \neq 0$ [5]. In eq. (B.1) the masses are taken to be pairwise grouped with opposite signs exactly as in the integral (2.7). The limit $\mathcal{N} \rightarrow \infty$ is taken in the two Random Matrix Theory partition functions so that $\mu_i \equiv m_i \pi \rho(0) \mathcal{N}$ and $\mu_i \equiv m_i \pi \rho(0) 2\mathcal{N}$, respectively, are kept fixed. In this limit the partition functions $\mathcal{Z}_{\text{QCD}_3}^{(2N_f)}$ and $\tilde{\mathcal{Z}}_{\text{QCD}_3}^{(2N_f)}$ become proportional to each other, with a μ_i -independent proportionality constant [2]. Similarly, also the partition functions $\mathcal{Z}_\nu^{(N_f)}$ and $\tilde{\mathcal{Z}}_\nu^{(N_f)}$ become proportional [4]. Below we derive relations between the unitary matrix integrals (2.7) and (2.12) from their Random Matrix representation (B.1) and (B.2), respectively. The key ingredient is to use the standard orthogonal polynomial technique of Random Matrix Theory, and next rewrite all expressions in terms of the partition functions themselves [40, 8]. Additional details can be found in ref. [8].

We begin by expressing the integrals (B.1) and (B.2) in terms of eigenvalue representations. This permits us to consider non-integer values of ν through analytical continuation. The starting point is a relation between the orthogonal polynomials of chUE (QCD₄) and those of the UE (QCD₃):

$$P_{l, \text{chUE}}^{(N_f, \nu=-1/2)}(\{m_i\}; z^2) = P_{2l, \text{UE}}^{(2N_f)}(\{m_i\}; z) . \quad (\text{B.4})$$

This simple identity is easily derived from the definition of the two Random Matrix Theories involved [39]. Since in ref. [39] no explicit use was made of the measure, eq. (B.4) also holds in the massive case. On the right hand side the same N_f masses appear, but doubled into pairs with opposite sign.

Instead of using directly the orthogonal polynomials $P_l(\{m_i\}; \lambda)$, it is convenient to work with the “wave functions”,

$$\Psi_l(\{m_i\}; \lambda) \equiv \sqrt{\omega(\{m_i\}, \lambda)} P_l(\{m_i\}; \lambda) , \quad (\text{B.5})$$

where $\omega(\{m_i\}; \lambda)$ is the measure function (so that the wave functions $\Psi_l(\{m_i\}; \lambda)$ are orthogonal with respect to a weight of unity). It follows from eq. (3.11) of ref. [39] that one can use the wave functions of the UE inside the Christoffel-Darboux identity for the chUE kernel. We then find the following identity [35]:

$$\begin{aligned}
K_{\mathcal{N}, \text{chUE}}^{(N_f, \nu=-1/2)}(z^2, w^2) &= \frac{c_{2\mathcal{N}}^{(2N_f)}}{z^2 - w^2} \left(\Psi_{2\mathcal{N}, \text{UE}}^{(2N_f)}(z) w \Psi_{2\mathcal{N}-1, \text{UE}}^{(2N_f)}(w) - z \Psi_{2\mathcal{N}-1, \text{UE}}^{(2N_f)}(z) \Psi_{2\mathcal{N}, \text{UE}}^{(2N_f)}(w) \right) \\
&= \frac{1}{2} \frac{c_{2\mathcal{N}}^{(2N_f)}}{(z-w)} \left(\Psi_{2\mathcal{N}, \text{UE}}^{(2N_f)}(z) \Psi_{2\mathcal{N}-1, \text{UE}}^{(2N_f)}(w) - \Psi_{2\mathcal{N}-1, \text{UE}}^{(2N_f)}(z) \Psi_{2\mathcal{N}, \text{UE}}^{(2N_f)}(w) \right) \\
&\quad - \frac{1}{2} \frac{c_{2\mathcal{N}}^{(2N_f)}}{(z+w)} \left(\Psi_{2\mathcal{N}, \text{UE}}^{(2N_f)}(z) \Psi_{2\mathcal{N}-1, \text{UE}}^{(2N_f)}(w) + \Psi_{2\mathcal{N}-1, \text{UE}}^{(2N_f)}(z) \Psi_{2\mathcal{N}, \text{UE}}^{(2N_f)}(w) \right) \\
&= K_{2\mathcal{N}, \text{UE}}^{(2N_f)}(z, w) + K_{2\mathcal{N}, \text{UE}}^{(2N_f)}(-z, w), \tag{B.6}
\end{aligned}$$

where for clarity we have not explicitly indicated the mass dependence of the wave functions. The coefficients $c_{2\mathcal{N}}^{(2N_f)}$ are defined in ref. [39]. We have also used that the parity of the polynomials $P_l(\lambda)$ for the UE is equal to $(-1)^l$. The different kernels can all be expressed in terms the corresponding finite volume partition functions [40, 8], as we will see below. In fact, the relation (B.6) simply expresses Lemma I of Appendix A, divided on each side by the result of Theorem I.

In order to arrive at a more general result, we will also derive an equation involving $\mathcal{Z}_{\nu=+1/2}^{(N_f)}(\{\mu\})$. To do so we start with the kernel $K_{\mathcal{N}, \text{chUE}}^{(N_f, \nu=+1/2)}(z^2, w^2)$ with $\nu = +1/2$ in contrast to eq. (B.6). Here, we will make use of the following relation among the wavefunctions of the UE:

$$\Psi_{2l, \text{UE}}^{(2N_f, 2\alpha+2)}(z) = \text{sign}(z) \Psi_{2l+1, \text{UE}}^{(2N_f, 2\alpha)}(z). \tag{B.7}$$

This relation was shown in [39] for any number of massless flavors 2α , but it can easily be generalized to include any additional number of $2N_f$ (massless or massive) flavors as well, since they are merely spectators. The sign function takes into account that the left hand side is an even function. Employing again the relation eq. (B.4) we obtain

$$\begin{aligned}
K_{\mathcal{N}, \text{chUE}}^{(N_f, \nu=+1/2)}(z^2, w^2) &= \frac{c_{2\mathcal{N}}^{(2N_f+2)} c_{2\mathcal{N}-1}^{(2N_f+2)}}{z^2 - w^2} \left(\Psi_{2\mathcal{N}, \text{UE}}^{(2N_f+2)}(z) \Psi_{2\mathcal{N}-2, \text{UE}}^{(2N_f+2)}(w) - \Psi_{2\mathcal{N}, \text{UE}}^{(2N_f+2)}(w) \Psi_{2\mathcal{N}-2, \text{UE}}^{(2N_f+2)}(z) \right) \\
&= \frac{c_{2\mathcal{N}}^{(2N_f+2)} c_{2\mathcal{N}-1}^{(2N_f+2)}}{z^2 - w^2} \text{sign}(z) \text{sign}(w) \\
&\quad \left(\Psi_{2\mathcal{N}+1, \text{UE}}^{(2N_f)}(z) \Psi_{2\mathcal{N}-1, \text{UE}}^{(2N_f)}(w) - \Psi_{2\mathcal{N}+1, \text{UE}}^{(2N_f)}(w) \Psi_{2\mathcal{N}-1, \text{UE}}^{(2N_f)}(z) \right) \\
&= \frac{c_{2\mathcal{N}}^{(2N_f+2)} c_{2\mathcal{N}-1}^{(2N_f+2)}}{c_{2\mathcal{N}}^{(2N_f)}} \frac{1}{z^2 - w^2} \text{sign}(z) \text{sign}(w) \\
&\quad \left(\Psi_{2\mathcal{N}+1, \text{UE}}^{(2N_f)}(z) w \Psi_{2\mathcal{N}, \text{UE}}^{(2N_f)}(w) - \Psi_{2\mathcal{N}+1, \text{UE}}^{(2N_f)}(w) z \Psi_{2\mathcal{N}, \text{UE}}^{(2N_f)}(z) \right). \tag{B.8}
\end{aligned}$$

In the first step we have used the relation (B.7) and in the second step the recursion relation $\lambda P_l(\lambda) = c_{l+1} P_{l+1}(\lambda) + c_l P_{l-1}(\lambda)$. With the same rewriting as in eq. (B.6) we arrive at

$$\begin{aligned}
K_{\mathcal{N}, \text{chUE}}^{(N_f, \nu=+1/2)}(z^2, w^2) &= \frac{c_{2\mathcal{N}}^{(2N_f+2)} c_{2\mathcal{N}-1}^{(2N_f+2)}}{c_{2\mathcal{N}}^{(2N_f)}} \text{sign}(z) \text{sign}(w) \\
&\quad \frac{1}{2} \left[\frac{1}{(z-w)} \left(\Psi_{2\mathcal{N}+1, \text{UE}}^{(2N_f)}(z) \Psi_{2\mathcal{N}, \text{UE}}^{(2N_f)}(w) - \Psi_{2\mathcal{N}, \text{UE}}^{(2N_f)}(z) \Psi_{2\mathcal{N}+1, \text{UE}}^{(2N_f)}(w) \right) \right. \\
&\quad \left. + \frac{1}{(z+w)} \left(-\Psi_{2\mathcal{N}+1, \text{UE}}^{(2N_f)}(z) \Psi_{2\mathcal{N}, \text{UE}}^{(2N_f)}(w) - \Psi_{2\mathcal{N}, \text{UE}}^{(2N_f)}(z) \Psi_{2\mathcal{N}+1, \text{UE}}^{(2N_f)}(w) \right) \right] \tag{B.9}
\end{aligned}$$

$$= \text{sign}(z)\text{sign}(w) \frac{c_{2\mathcal{N}}^{(2N_f+2)} c_{2\mathcal{N}-1}^{(2N_f+2)}}{c_{2\mathcal{N}}^{(2N_f)} c_{2\mathcal{N}+1}^{(2N_f)}} \left(K_{2\mathcal{N}+1, \text{UE}}^{(2N_f)}(z, w) - K_{2\mathcal{N}+1, \text{UE}}^{(2N_f)}(-z, w) \right).$$

In the large- \mathcal{N} limit the mass dependence of the recursion coefficients $c_{2\mathcal{N}}^{(2N_f)}$ becomes subleading and these coefficients simply cancel out. Again, this relation has a direct interpretation in terms of finite volume partition functions: it is Lemma II of Appendix B divided on each side by the relation of Theorem I.

Adding and subtracting eqs. (B.6) and (B.9) we finally arrive at

$$K_{\text{UE}}^{(2N_f)}(\{\mu_i\}, \pm\zeta, \omega) = \frac{1}{2} \left(K_{\text{chUE}}^{(N_f, \nu=-1/2)}(\{\mu_i\}, \zeta^2, \omega^2) \pm K_{\text{chUE}}^{(N_f, \nu=+1/2)}(\{\mu_i\}, \zeta^2, \omega^2) \right) \quad (\text{B.10})$$

in the microscopic large- \mathcal{N} limit, for $\zeta, \omega \geq 0$. One of these (the one with “+” signs) was previously known only in the massless case, where it can readily be derived from the generalized Laguerre ensemble [47].

We now make the following observation. From the kernel relation (B.10) it follows that all spectral correlation functions of the two Random Matrix Theories (B.1) and (B.2) are related in a fairly simple way. If one believes that these relations extend to the microscopic spectral correlators of the Dirac operators in QCD₃ and QCD₄, then one obtains all spectral correlation functions $\rho_{\text{QCD}_3}^{(2N_f)}(\zeta_1, \dots, \zeta_k; \{\mu_i\})$ in QCD₃ from the kernel relevant for QCD₄. The most obvious relation that follows from (B.10) is the one for the microscopic spectral densities,

$$\rho_{\text{QCD}_3}^{(2N_f)}(\zeta; \{\mu_i\}) = \frac{1}{2} \left[\rho_{\text{QCD}_4}^{(N_f, \nu=+1/2)}(\zeta; \{\mu_i\}) + \rho_{\text{QCD}_4}^{(N_f, \nu=-1/2)}(\zeta; \{\mu_i\}) \right]. \quad (\text{B.11})$$

This relation is here derived within the context of Random Matrix Theory. As shown in this paper, it also follows directly from the field theory formulation, through the replica method.

We note that based on the exact relationship of two-point correlation functions to the Random Matrix Theory partition functions [40, 8] (which in turn are proportional to the finite-volume field theory partition functions), eq. (B.10) implies the following identity:

$$\frac{\mathcal{Z}_{\text{QCD}_3}^{(2N_f+2)}(\{\mu_i\}, \pm\zeta, \omega)}{\mathcal{Z}_{\text{QCD}_3}^{(2N_f)}(\{\mu_i\})} = \pi(\zeta\omega)^{1/2} \left(\frac{\mathcal{Z}_{\nu=-1/2}^{(N_f+2)}(\{\mu_i\}, \zeta, \omega)}{\mathcal{Z}_{\nu=-1/2}^{(N_f)}(\{\mu_i\})} \mp \frac{\mathcal{Z}_{\nu=+1/2}^{(N_f+2)}(\{\mu_i\}, \zeta, \omega)}{\mathcal{Z}_{\nu=+1/2}^{(N_f)}(\{\mu_i\})} \right). \quad (\text{B.12})$$

This particular relation is simply the identity of Theorem III divided on each side by the identity of Theorem I, both of which were proven in section 4.1. The right hand side involves a sum or difference of two different ratios of partition functions that cannot easily be combined. The identity as it stands is therefore not suitable for deriving, by means of the replica method, identities among spectral correlators of the Dirac operators in the two theories. We note, however, that in the limit, $\zeta, \omega \rightarrow 0$ it is evidently consistent with the general identity (3.12). In this limit the second term on the right hand side of (B.12) vanishes. Although the first term is also multiplied by $(\zeta\omega)^{1/2}$, the product of the two remains finite. Indeed, invoking *flavor-topology duality* [38], we can express the product in terms of a partition function with two massless particle less, at the cost of increasing ν ,

$$\frac{\mathcal{Z}_{\text{QCD}_3}^{(2N_f+2)}(\{\mu_i\}, 0, 0)}{\mathcal{Z}_{\text{QCD}_3}^{(2N_f)}(\{\mu_i\})} = 2 \prod_{j=1}^{N_f} \frac{1}{\mu_j^2} \frac{\mathcal{Z}_{\nu=+3/2}^{(N_f)}(\{\mu_i\})}{\mathcal{Z}_{\nu=-1/2}^{(N_f)}(\{\mu_i\})}. \quad (\text{B.13})$$

By applying flavor-topology duality, the relation (B.13) also follows from the more general identity (3.12) given in Theorem I alone. To see this, we take the ratio of eq. (3.12) for $N_f + 1$ and N_f and then send the extra mass to zero. The factors $\mathcal{Z}_{+1/2}^{(N_f)}(\{\mu_i\})$ cancel after the flavor-topology shift (3.11) on the partition functions with $\nu = -1/2$ and $\nu = +1/2$ and we arrive at eq. (B.13).

C Partition Function and Spectral Sum Rules to Sixth Order

In this Appendix we give the terms of order μ^5 and μ^6 in the small expansion of the QCD₃ partition function $\mathcal{Z}_q^{(N)}$, The terms up to order μ^4 were already given in eq. (5.13). For the terms of $O(\mu^4)$ we find

$$\begin{aligned} \mathcal{Z}_q^{(N)}|_{\mu^5} = & -\frac{2q(N^2 - q^2)(20 - 3N^2 + 7q^2)}{5N(N^2 - 1)(N^2 - 4)(N^2 - 9)(N^2 - 16)}\text{Tr}M^5 \\ & - \frac{q(N^2 - q^2)[N^4 + N^2(4 - 5q^2) + 24(q^2 - 4)]}{4N^2(N^2 - 1)(N^2 - 4)(N^2 - 9)(N^2 - 16)}\text{Tr}M^4\text{Tr}M \\ & + \frac{q(N^2 - q^2)(N^2q^2 + 12q^2 - N^4 + 12N^2 - 48)}{3N^2(N^2 - 1)(N^2 - 4)(N^2 - 9)(N^2 - 16)}\text{Tr}M^3\text{Tr}M^2 \\ & + \frac{q(N^2 - q^2)(4 - q^2)}{3N(N^2 - 1)(N^2 - 4)(N^2 - 16)}\text{Tr}M^3(\text{Tr}M)^2 \\ & + \frac{q(N^2 - q^2)(N^4 - N^2q^2 - 18N^2 + 2q^2 + 88)}{8N(N^2 - 1)(N^2 - 4)(N^2 - 9)(N^2 - 16)}\text{Tr}M(\text{Tr}M^2)^2 \\ & + \frac{q(N^2 - q^2)(N^4q^2 - 14N^2q^2 + 24q^2 - 3N^4 + 40N^2 - 96)}{12N^2(N^2 - 1)(N^2 - 4)(N^2 - 9)(N^2 - 16)}\text{Tr}M^2(\text{Tr}M)^3 \\ & + \frac{q[N^4(q^4 - 10q^2 + 15) - 20N^2(q^4 - 9q^2 + 11) + 6(13q^4 - 100q^2 + 96)]}{120N(N^2 - 1)(N^2 - 4)(N^2 - 9)(N^2 - 16)}(\text{Tr}M)^5. \end{aligned} \quad (\text{C.1})$$

At order μ^6 we find

$$\begin{aligned} \mathcal{Z}_q^{(N)}|_{\mu^6} = & \sum_{l=0}^5 A_l \text{Tr}M^{6-l}(\text{Tr}M)^l + \sum_{k=1}^2 B_k \text{Tr}M^{6-2k}(\text{Tr}M^2)^k \\ & + \sum_{i=1}^3 C_i \left(\text{Tr}M^i \text{Tr}M^{3-i}\right)^2 + D \left(\text{Tr}M \text{Tr}M^2 \text{Tr}M^3\right). \end{aligned} \quad (\text{C.2})$$

where

$$\begin{aligned} A_0 &= \frac{(N^2 - q^2)[(N^2 - 4)(N^2 - 16) + 140q^2 - 14N^2q^2 + 21q^4]}{3N(N^2 - 1)(N^2 - 4)(N^2 - 9)(N^2 - 16)(N^2 - 25)}, \\ A_1 &= \frac{2(N^2 - q^2)[-5(N^2 - 4)(N^2 - 16) + q^2(-200 - 25N^2 + 3N^4 + 70q^2 - 7N^2q^2)]}{5N^2(N^2 - 1)(N^2 - 4)(N^2 - 9)(N^2 - 16)(N^2 - 25)}, \\ A_2 &= \frac{(1 - q^2)(N^2 - q^2)[(N^2 - 4)(N^2 + 23) + 13q^2 - 5N^2q^2]}{8N(N^2 - 1)^2(N^2 - 4)(N^2 - 9)(N^2 - 25)}, \\ A_3 &= \frac{-(N^2 - q^2)}{18N^2(N^2 - 1)^2(N^2 - 4)(N^2 - 9)(N^2 - 16)(N^2 - 25)} \left[(N^2 - 4)(N^2 - 16)(9N^2 + 15) \right. \\ & \quad \left. - 1541N^2q^2 + 356N^4q^2 - 15N^6q^2 - 60q^4 + 229N^2q^4 - 51N^4q^4 + 2N^6q^4 \right], \end{aligned}$$

$$\begin{aligned}
A_4 &= \frac{(N^2 - q^2) [3(N^2 - 4)(N^2 - 16) + q^2(-280 + 136N^2 - 6N^4 + 38q^2 - 24N^2q^2 + N^4q^2)]}{48N(N^2 - 1)^2(N^2 - 4)(N^2 - 16)(N^2 - 25)}, \\
A_5 &= \left[720N^2(N^2 - 1)^2(N^2 - 4)(N^2 - 9)(N^2 - 16)(N^2 - 25) \right]^{-1} \left[15N^2(4 - N^2)(N^2 - 9)(N^2 - 16) \right. \\
&\quad + 5760q^2 - 29416N^2q^2 + 13756N^4q^2 - 1545N^6q^2 + 45N^8q^2 - 2400q^4 + 14290N^2q^4 \\
&\quad \left. - 5850N^4q^4 + 575N^6q^4 - 15N^8q^4 + 240q^6 - 1258N^2q^6 + 458N^4q^6 - 41N^6q^6 + N^8q^6 \right], \\
B_1 &= \frac{-(N^2 - q^2)}{8N^2(N^2 - 1)^2(N^2 - 4)(N^2 - 9)(N^2 - 16)(N^2 - 25)} \left[(40 - 17N^2 + N^4)(N^2 - 4)(N^2 - 16) \right. \\
&\quad \left. + q^2(-800 - 484N^2 + 90N^4 - 6N^6 + 40q^2 + 75N^2q^2 + 5N^4q^2) \right], \\
B_2 &= \frac{(N^2 - q^2)}{48N(N^2 - 1)^2(N^2 - 4)(N^2 - 9)(N^2 - 16)(N^2 - 25)} \left[(N^4 - 27N^2 + 98)(N^2 - 4)(N^2 - 16) \right. \\
&\quad \left. + q^2(-3480 - 164N^2 + 46N^4 - 2N^6 + 358q^2 + N^2q^2 + N^4q^2) \right], \\
C_1 &= C_2 = \frac{(N^2 - q^2)(q^2 - 1) [(N^4 - 15N^2 - 10)(N^2 - 4) + q^2(-10 + 3N^2 - N^4)]}{32N^2(N^2 - 1)^2(N^2 - 4)(N^2 - 9)(N^2 - 25)}, \\
C_3 &= \frac{2(N^2 - q^2)}{9N^4(N^2 - 1)^2(N^2 - 4)(N^2 - 9)(N^2 - 16)(N^2 - 25)} \left[(15 - 3N^2)(N^2 - 4)(N^2 - 16) \right. \\
&\quad \left. + q^2(-600 + 3N^2 - 4N^4 + N^6 + 90q^2 - 29N^2q^2 - N^4q^2) \right], \\
D &= \frac{(N^2 - q^2) [3(N^2 - 4)(N^2 - 16) - q^2(180 - 32N^2 + 2N^4 - 13q^2 - 2N^2q^2)]}{6N(N^2 - 1)^2(N^2 - 4)(N^2 - 16)(N^2 - 25)}. \tag{C.3}
\end{aligned}$$

From (C.2) and (C.3) we derive the following additional spectral sum rules for an **even** number of Dirac eigenvalues:

$$\begin{aligned}
\left\langle \sum_n \frac{1}{\zeta_n^6} \right\rangle &= 6 A_0 \quad , \quad \left\langle \sum_n \frac{1}{\zeta_n} \sum_m \frac{1}{\zeta_m^5} \right\rangle = -5 A_1, \\
\left\langle \left(\sum_n \frac{1}{\zeta_n} \right)^4 \sum_m \frac{1}{\zeta_m^2} \right\rangle &= 48 A_4 \quad , \quad \left\langle \sum_n \frac{1}{\zeta_n} \sum_m \frac{1}{\zeta_m^2} \sum_p \frac{1}{\zeta_p^3} \right\rangle = 6 D, \\
\left\langle \left(\sum_n \frac{1}{\zeta_n} \right)^6 \right\rangle &= -6! A_5 \quad , \quad \left\langle \left(\sum_n \frac{1}{\zeta_n} \right)^2 \sum_m \frac{1}{\zeta_m^4} \right\rangle = 8 A_2, \\
\left\langle \left(\sum_n \frac{1}{\zeta_n} \right)^3 \sum_m \frac{1}{\zeta_m^3} \right\rangle &= -18 A_3 \quad , \quad \left\langle \left(\sum_n \frac{1}{\zeta_n} \right)^2 \left(\sum_m \frac{1}{\zeta_m^2} \right)^2 \right\rangle = -32 C_1, \\
\left\langle \left(\sum_n \frac{1}{\zeta_n^3} \right)^2 \right\rangle &= -18 N^2 C_3 \quad , \quad \left\langle \left(\sum_n \frac{1}{\zeta_n^2} \right)^3 \right\rangle = 48 B_2, \\
\left\langle \sum_n \frac{1}{\zeta_n^2} \sum_m \frac{1}{\zeta_m^4} \right\rangle &= -8 B_1. \tag{C.4}
\end{aligned}$$

We remind the reader that $q = 0, 1$ for even and odd number of flavors respectively.

References

- [1] R.D. Pisarski, Phys. Rev. **D29** (1984) 2423.
- [2] J.J.M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **73** (1994) 2288.
- [3] P.H. Damgaard, U.M. Heller, A. Krasnitz and T. Madsen, Phys. Lett. **B440** (1998) 129.
- [4] E.V. Shuryak and J.J.M. Verbaarschot, Nucl. Phys. **A560** (1993) 306.
J.J.M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **70** (1993) 3852.
- [5] G. Akemann, P.H. Damgaard, U. Magnea and S. Nishigaki, Nucl. Phys. **B487** (1997) 721.
- [6] P.H. Damgaard and S.M. Nishigaki, Phys. Rev. **D57** (1998) 5299.
- [7] J. Christiansen, Nucl. Phys. **B547** (1999) 329.
- [8] G. Akemann and P.H. Damgaard, Nucl. Phys. **B528** (1998) 411; Phys. Lett. **B432** (1998) 390.
- [9] U. Magnea, Phys. Rev. **D61** (2000) 056005; Phys. Rev. **D62** (2000) 016005; hep-th/0009208.
- [10] J.J.M. Verbaarschot, Phys. Rev. Lett. **72** (1994) 2531.
- [11] C. Hilmoine and R. Niclasen, Phys. Rev. **D62** (2000) 096013.
- [12] T. Nagao and S.M. Nishigaki, hep-th/0005077.
- [13] J.C. Osborn, D. Toublan and J.J.M. Verbaarschot, Nucl. Phys. **B540** (1999) 317.
P.H. Damgaard, J.C. Osborn, D. Toublan and J.J.M. Verbaarschot, Nucl. Phys. **B547** (1999) 305.
- [14] D. Toublan and J.J.M. Verbaarschot, hep-th/0012144.
- [15] E. Brézin, Lect. Notes Phys. **216** (1980) 113.
- [16] K.B. Efetov, Adv. Phys. 32 (1983) 53; *Supersymmetry in disorder and chaos* (Cambridge University Press, Cambridge, 1997).
- [17] J.J.M. Verbaarschot, H.A. Weidenmüller, and M.R. Zirnbauer, Phys. Rep. **129** (1985) 367.
- [18] A. Altland and B. D. Simons, Nucl. Phys. **B562** (1999) 445.
- [19] K. Takahashi and S. Iida, Nucl. Phys. **B573** (2000) 685.
- [20] T. Guhr, T. Wilke and H. A. Weidenmüller, Phys. Rev. Lett. **85** (2000) 2252.
- [21] C. Bernard and M. Golterman, Phys. Rev. **D49** (1994) 486; hep-lat/9311070.
M.F.L. Golterman, Acta Phys. Polon. **B25** (1994).
- [22] M.F.L. Golterman and K.C. Leung, hep-lat/9711033.
- [23] S. Sharpe, Phys. Rev. **D46** (1992) 3146.
- [24] S.F. Edwards and P.W. Anderson, J. Phys. **F5** (1975) 865.
- [25] J.J.M. Verbaarschot and M.R. Zirnbauer, J. Phys. **A17** (1985) 1093.
- [26] A. Kamenev and M. Mézard, J. Phys. **A32** (1999) 4373; Phys. Rev. **B60** (1999) 3944.

- [27] I.V. Yurkevich and I.V. Lerner, Phys. Rev. **B60** (1999) 3955.
- [28] E. Kanieper, cond-mat/9908130.
- [29] M.R. Zirnbauer, cond-mat/9903338.
- [30] P.H. Damgaard and K. Splittorff, Nucl. Phys. **B572** (2000) 478; Phys. Rev. **D62** (2000) 054509.
P.H. Damgaard, Phys. Lett. **B476** (2000) 465.
- [31] D. Dalmazi and J.J.M. Verbaarschot, hep-th/0005229.
- [32] M.E. Berbenni-Bitsch, M. Göckeler, H. Hehl, S. Meyer, P.E.L. Rakow, A. Schäfer and T. Wettig,
Phys. Lett. **B 466** (1999) 293; hep-lat/9908043.
- [33] R. Szabo, hep-th/0009237.
- [34] A.N. Redlich, Phys. Rev. Lett. **52** (1984) 18.
- [35] G. Akemann and P.H. Damgaard, Nucl. Phys. **B576** (2000) 597.
- [36] C. Itzykson and J.-B. Zuber, J. Math. Phys. **21** (1980) 411.
- [37] R. Brower, P. Rossi and C.-I. Tan, Nucl. Phys. **B190** [FS3] (1981) 699.
A.D. Jackson, M.K. Şener and J.J.M. Verbaarschot, Phys. Lett. **B387** (1996) 355.
A.B. Balantekin, Phys. Rev. **D62** (2000) 085017.
- [38] J.J.M. Verbaarschot, Nato Lectures Cambridge 1997 on “Confinement, Duality, and Nonpertur-
bative Aspects of QCD” p. 343-378, hep-th/9710114.
T. Akuzawa and M. Wadati, J. Phys. Soc. Jap. **67** (1998) 2151.
- [39] G. Akemann, P.H. Damgaard, U. Magnea and S. Nishigaki, Nucl. Phys. **B519** (1998) 628.
- [40] P.H. Damgaard, Phys. Lett. **B424** (1998) 322.
- [41] D.J. Gross and M. Newmann, Nucl. Phys. **B380** (1992) 168.
- [42] A. Mironov, A. Morozov and G. Semenoff, Int. J. Mod. Phys. **A11** (1996) 5031.
- [43] S. Samuel, J. Math. Phys. **21** (1980) 2695.
- [44] P. Zinn-Justin, Nucl. Phys. **B497** (1997) 725.
- [45] B. de Wit and G. 't Hooft, Phys. Lett. **B69** (1977) 61.
- [46] P. Zinn-Justin, Commun. Math. Phys. **194** (1998) 631.
- [47] C. Hilmoine, M.Sc. Thesis, The Niels Bohr Institute (2000), (unpublished).
- [48] J.J. Duistermaat and G. Heckman, Invent. Math. 69 (1982) 259; *ibid.* 72 (1983) 153.
- [49] R.J. Szabo, Lect. Notes Phys. M63 (2000) 1.